UNIT 1C

Elementary Matrices and Special Matrices

Unit 1C · Sections 1.5 – 1.7

MATH 244 — Linear Algebra
Elementary Linear Algebra, Anton & Rorres, 11th edition
King Saud University · Department of Mathematics

LEARNING OUTCOMES

What you should be able to do

Section 1.5

- Reverse any elementary row operation, and decide whether two matrices are row equivalent.
- Recognise an elementary matrix, and write down the elementary matrix that performs a given row operation.
- Use the Equivalence Theorem to move between invertibility, $Ax = 0$, the reduced row echelon form, and products of elementary matrices.
- Run the inversion algorithm $\mathrm{RREF}([A \mid I]) = \left[I \mid A^{-1}\right]$, and stop early when a row of zeros appears.

Section 1.6

- Explain why a linear system has zero, one, or infinitely many solutions — never exactly two.
- Solve $Ax = b$ by $x = A^{-1}b$, and solve several systems with a common coefficient matrix at once.
- Use a one-sided inverse, and the six equivalent statements, as invertibility tests.
- Find the conditions on b that make $Ax = b$ consistent.

Section 1.7

- Compute inverses, powers and products involving diagonal matrices by inspection.
- Use the properties of triangular and symmetric matrices, including AA^T and A^TA.

SECTION 1.5

Elementary Matrices and a Method for Finding A^{-1}

GOAL — Turn a row operation into a matrix and turn that idea into an algorithm that both tests invertibility and produces the inverse.

1. Reversing a row operation; row equivalence
2. Elementary matrices
3. Row operations as multiplication on the left
4. Every elementary matrix is invertible
5. The Equivalence Theorem — four statements
6. The inversion algorithm $\mathrm{RREF}([A \mid I]) = [I \mid A^{-1}]$

RECALL

The three elementary row operations

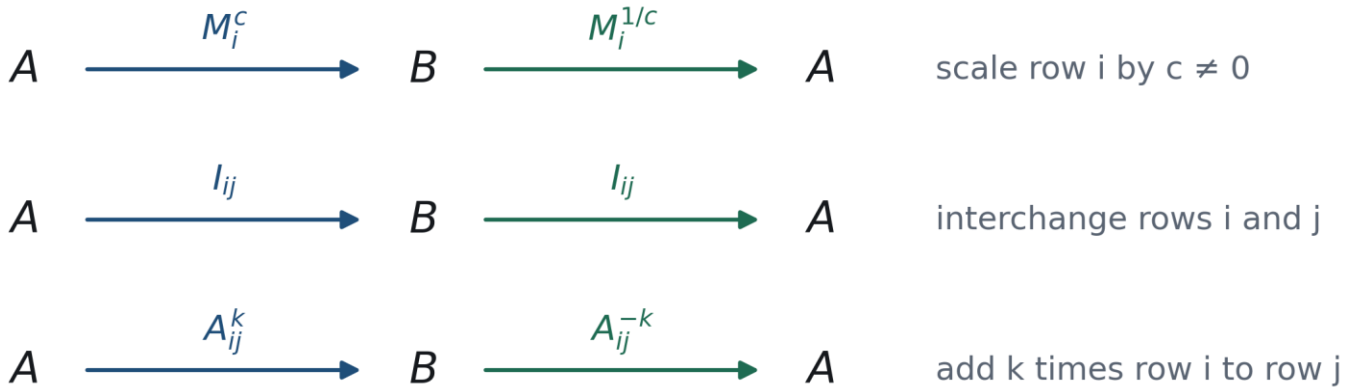
- From Section 1.1, and written throughout MATH 244 in the course notation:

	What it does	Course symbol	Undone by
Interchange	swap rows i and j	$I_{i,j}$	$I_{i,j}$
Scale	multiply row i by $c \neq 0$	$M_i^{(c)}$	$M_i^{(1/c)}$
Add	add k times row i to row j	$A_{i,j}^{(k)}$	$A_{i,j}^{(-k)}$

- In $A_{i,j}^{(k)}$ the FIRST index is the row you take a multiple of; the SECOND is the row that actually changes.

KEY IDEA

Every row operation is reversible



> Because each operation can be undone, a SEQUENCE of operations can be undone too — apply the inverse operations in the reverse order. That single observation is what makes the next definition symmetric.

DEFINITION 1

Row equivalence

> Matrices A and B are ROW EQUIVALENT if either (hence each) can be obtained from the other by a sequence of elementary row operations.

- Reversibility is exactly what lets us say “either, hence each”. Row equivalence is therefore an equivalence relation: it is reflexive, symmetric and transitive.
- Two matrices in the same class have the SAME reduced row echelon form, and — because that form is unique — each class contains exactly one reduced row echelon matrix.

TECHNIQUE

Testing row equivalence

Reduce both, then compare

- A and B are row equivalent $\Leftrightarrow \mathrm{RREF}(A) = \mathrm{RREF}(B)$.

Example — are these two row equivalent?

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathrm{RREF}(B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The two reduced forms differ, so A and B are NOT row equivalent. No amount of searching for a sequence of operations was needed.

DEFINITION 2

Elementary matrix

> A matrix E is called an ELEMENTARY MATRIX if it can be obtained from an identity matrix by performing a SINGLE elementary row operation.

Example 1 — four elementary matrices and the operations that produce them

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Left to right: $M_2^{(-3)}$ on I_2; $I_{2,4}$ on I_4; $A_{3,1}^{(3)}$ on I_3; $M_1^{(1)}$ on I_3 — the identity itself counts, since multiplying a row by 1 is a legal operation.

Every elementary matrix of size 2×2

- There are only three kinds, so the whole family fits on one slide. Here $c \neq 0$ and k is any scalar.

$$M_1^{(c)} = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \qquad M_2^{(c)} = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$$

$$I_{1,2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{1,2}^{(k)} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \qquad A_{2,1}^{(k)} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

- Read the pattern: an ADD operation puts the multiplier OFF the diagonal, in the row that changes; a SCALE operation puts it ON the diagonal.

THEOREM 1.5.1

Row operations by matrix multiplication

> If the elementary matrix E results from performing a certain row operation on I_m, and if A is an $m \times n$ matrix, then EA is the matrix that results when that SAME row operation is performed on A.

Example 2

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}, \qquad EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

- E is I_3 with 3 times row 1 added to row 3, i.e. $A_{1,3}^{(3)}$ — and EA is exactly A with 3 times row 1 added to row 3.

This theorem is a proof tool, not a computing tool. In practice you perform the row operation directly rather than building E and multiplying.

TABLE

Undoing an elementary matrix

- The right-hand column is the INVERSE OPERATION of the left. Applying it to E rebuilds I.

	Row operation on I that produces E	**Row operation on E that reproduces I**
Scale	multiply row i by $c \neq 0$	multiply row i by $1/c$
Interchange	interchange rows i and j	interchange rows i and j
Add	add k times row i to row j	add $-k$ times row i to row j

- Interchange is its own inverse. That is worth remembering: $\left(I_{i,j}\right)^{-1} = I_{i,j}$.

THEOREM 1.5.2

Every elementary matrix is invertible

> Every elementary matrix is invertible, and its inverse is again an elementary matrix.

Proof

Let E come from I by some row operation and let E_0 come from I by the INVERSE operation. Applying Theorem 1.5.1 twice, and using that inverse operations cancel each other,

$$E_0E = I \quad \text{and} \quad EE_0 = I$$

so $E_0 = E^{-1}$, and E_0 is elementary by construction. ■

Example 3

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},\ E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix},\ E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

THEOREM 1.5.3

Equivalent Statements

- If A is an $n \times n$ matrix, the following are equivalent — all true, or all false.

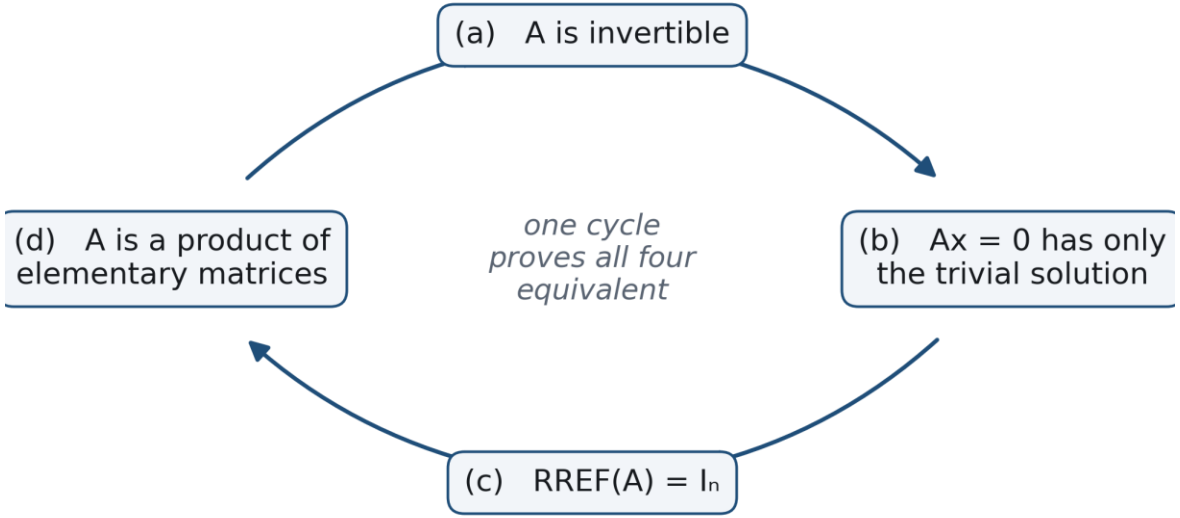
> (a) A is invertible. (b) $Ax = 0$ has only the trivial solution. (c) The reduced row echelon form of A is I_n.
> (d) A is expressible as a product of elementary matrices.

- This is the first of several such lists. As the course develops, more statements are ADDED to it — determinants in Chapter 2, rank and column space in Chapter 4 — and the list becomes the main organising theorem of linear algebra.

Note the word SQUARE. Every statement here is false for a non-square matrix, because invertibility is not even defined.

PROOF

One cycle proves everything

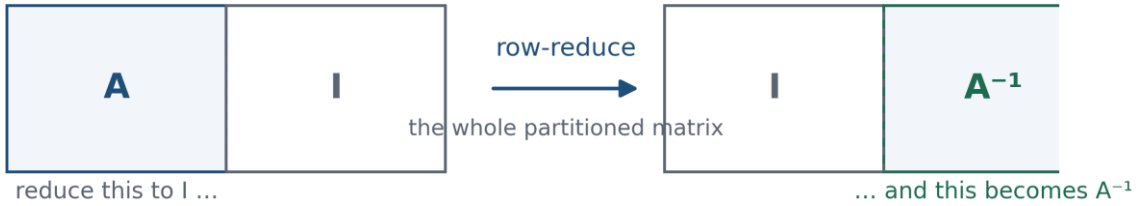


- $(a) \Rightarrow (b)$: multiply $Ax_0 = 0$ by A^{-1}. $(b) \Rightarrow (c)$: only the trivial solution forces every column to be a pivot column. $(c) \Rightarrow (d)$: $E_k \cdots E_1 A = I_n$, so $A = E_1^{-1} \cdots E_k^{-1}$. $(d) \Rightarrow (a)$: a product of invertible matrices is invertible.

ALGORITHM

Inverting a matrix by row reduction

- From $E_k \cdots E_1 A = I_n$, multiplying on the right by A^{-1} gives $A^{-1} = E_k \cdots E_1 I_n$: the very same operations that carry A to I carry I to A^{-1}.



> $\mathrm{RREF}([A \mid I]) = [I \mid A^{-1}]$ — and if a row of zeros ever appears on the LEFT of the partition, stop: A is not invertible.

EXAMPLE 4

Using row operations to find A^{-1}

Adjoin the identity and reduce the left half

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \overset{A_{1,2}^{(-2)},\, A_{1,3}^{(-1)}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{A_{2,3}^{(2)}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \overset{M_3^{(-1)}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

- The forward phase is done: the left half is now upper triangular with 1s on the diagonal. Continue on the next slide.

The backward phase

$$\overset{A_{3,2}^{(3)},\ A_{3,1}^{(-3)}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\overset{A_{2,1}^{(-2)}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

The left half is I, so the right half is the inverse

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Always spend ten seconds checking one column of $AA^{-1} = I$ before moving on.

EXAMPLE 5

Showing that a matrix is NOT invertible

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \overset{A_{1,2}^{(-2)},\, A_{1,3}^{(1)}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

$$\overset{A_{2,3}^{(1)}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

- A row of zeros has appeared on the LEFT of the partition, so the left half can never become I. By parts (a) and (c) of Theorem 1.5.3, A is not invertible — and we stop here.

The right half is now meaningless. Do not report it as an inverse.

EXAMPLE 6

Reading a homogeneous system off the theorem

- A homogeneous system has only the trivial solution EXACTLY WHEN its coefficient matrix is invertible — that is parts (a) and (b) of Theorem 1.5.3.

$$\text{(a)}\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 + 8x_3 &= 0 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x_1 + 6x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

- System (a) has the coefficient matrix of Example 4, which is invertible: only the trivial solution.

- System (b) has the coefficient matrix of Example 5, which is not: it has nontrivial solutions, in fact infinitely many.

SECTION 1.6

More on Linear Systems and Invertible Matrices

GOAL — Use the inverse as a solving tool, settle how many solutions a system can have, and answer the question: for which right-hand sides is $Ax = b$ consistent?

1. Zero, one, or infinitely many solutions
2. Solving $Ax = b$ by matrix inversion
3. Several systems with a common coefficient matrix
4. A one-sided inverse is enough
5. The Equivalence Theorem — now six statements
6. A fundamental problem: when is $Ax = b$ consistent?

THEOREM 1.6.1

Zero, one, or infinitely many — nothing else

> A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Proof

Suppose $Ax = b$ has two distinct solutions $x_1 \neq x_2$, and put $x_0 = x_1 - x_2 \neq 0$. Then $Ax_0 = Ax_1 - Ax_2 = b - b = 0$, and for every scalar λ,

$$A(x_1 + \lambda x_0) = Ax_1 + \lambda(Ax_0) = b + \lambda\, 0 = b$$

So every one of the infinitely many vectors $x_1 + \lambda x_0$, $\lambda \in \mathbb{R}$, is a solution. Two solutions force infinitely many. ■

This is why Section 1.1 could draw only three pictures for two lines: parallel, crossing, or coincident.

THEOREM 1.6.2

Solving by matrix inversion

> If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix b the system $Ax = b$ has exactly one solution, namely $x = A^{-1}b$.

Proof

$A(A^{-1}b) = b$, so $A^{-1}b$ IS a solution. If x_0 is any solution then $Ax_0 = b$; multiplying on the left by A^{-1} gives $x_0 = A^{-1}b$. So it is the ONLY one. ■

Notice the two halves: existence comes from substituting, uniqueness from multiplying by A^{-1}. That pattern will recur all year.

EXAMPLE 1

Solving a system with A^{-1}

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17 \end{aligned} \quad \Leftrightarrow \quad Ax = b, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \; b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

- This is the matrix of Example 4 in Section 1.5, whose inverse we already computed.

$$x = A^{-1}b = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

- So $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

WARNING

When this method does and does not apply

The method of Example 1 applies only when the system has AS MANY EQUATIONS AS UNKNOWNS and the coefficient matrix is INVERTIBLE.

- It is also the slower method. Computing A^{-1} is essentially a Gauss–Jordan reduction of an $n \times 2n$ matrix, followed by a multiplication; Gauss–Jordan on $[A \mid b]$ alone answers the same question with less work.

> Use $x = A^{-1}b$ when the SAME A is reused with many different right-hand sides, or when you want a FORMULA for x in terms of b. Use Gauss–Jordan when you want one answer once.

TECHNIQUE

Several systems, one reduction

- Systems $Ax = b_1$, $Ax = b_2$, …, $Ax = b_k$ that share a coefficient matrix can be solved TOGETHER: augment A with all of the right-hand sides at once and reduce.

$$[\, A \mid b_1 \mid b_2 \mid \cdots \mid b_k \,]$$

> This costs one reduction instead of k, and — unlike $x = A^{-1}b$ — it still works when A is not invertible, or not even square. The last k columns just have to be read one at a time.

EXAMPLE 2

Solving two systems at once

$$\text{(a)}\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 + 8x_3 &= 9 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 + 8x_3 &= -6 \end{aligned}$$

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

- Fourth column: system (a) has $x = (1,0,1)$. Fifth column: system (b) has $x = (2,1,-1)$.

Reading a joint reduction

- You reduced $[\,A \mid b_1 \mid b_2\,]$ and obtained

$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 & 5 \end{array}\right]$$

A) $Ax = b_1$ has solution $(3,0,4)$ and $Ax = b_2$ has solution $(0,2,5)$.

B) $Ax = b_1$ has solution $(3,-1,4)$ and $Ax = b_2$ has solution $(0,2,5)$.

C) $Ax = b_1$ has solution $(3,-1,4)$ and $Ax = b_2$ is inconsistent.

D) There is not enough information; you must reduce $[\,A \mid b_2\,]$ separately.

THEOREM 1.6.3

A one-sided inverse is enough

> Let A be a square matrix. If B is square and $BA = I$, then $B = A^{-1}$. If B is square and $AB = I$, then $B = A^{-1}$.

Proof of the first half

Assume $BA = I$. If x_0 solves $Ax = 0$, multiply on the left by B: $BAx_0 = B\,0$, so $Ix_0 = 0$ and $x_0 = 0$. Hence $Ax = 0$ has only the trivial solution, and A is invertible by Theorem 1.5.3.

Now multiply $BA = I$ on the right by A^{-1}: $BAA^{-1} = IA^{-1}$, so $BI = A^{-1}$, i.e. $B = A^{-1}$. ■

In practice: to verify a candidate inverse you only have to check ONE of the two products.

THEOREM 1.6.4

Equivalent Statements — the list grows to six

- For an $n \times n$ matrix A, all of the following are equivalent.

(a) A is invertible.

(b) $Ax = 0$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $Ax = b$ is CONSISTENT for every $n \times 1$ matrix b.

(f) $Ax = b$ has EXACTLY ONE solution for every $n \times 1$ matrix b.

> The equivalence of (e) and (f) is worth pausing on: for a SQUARE matrix, showing that $Ax = b$ has at least one solution for every b is enough to conclude that it has exactly one.

THEOREM 1.6.5

Invertible products have invertible factors

> Let A and B be square matrices of the same size. If AB is invertible, then A and B must both be invertible.

Proof

If $Bx_0 = 0$ then $(AB)x_0 = A(Bx_0) = A\,0 = 0$, so $x_0 = 0$ because AB is invertible. Hence $Bx = 0$ has only the trivial solution and B is invertible.

Then $A = (AB)B^{-1}$ is a product of invertible matrices, so A is invertible too. ■

Together with $(AB)^{-1} = B^{-1}A^{-1}$ from Section 1.4, this says: a product of square matrices is invertible exactly when every factor is.

For which b is $Ax = b$ consistent?

> Fix an $m \times n$ matrix A. Find ALL $m \times 1$ matrices b for which the system $Ax = b$ is consistent.

- If A is square and invertible, Theorem 1.6.2 answers this completely: EVERY b works, and each gives one solution.
- If A is not square, or is square but not invertible, then b usually has to satisfy conditions. Those conditions are found by the elimination of Section 1.2 — carry the symbols $b_1, b_2, \ldots$ along in the last column.

This question is the seed of the column space in Chapter 4. Keep it in view.

EXAMPLE 3

A condition appears

$$x_1 + x_2 + 2x_3 = b_1$$
$$x_1 + x_3 = b_2$$
$$2x_1 + x_2 + 3x_3 = b_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array}\right] \xrightarrow{A_{1,2}^{(-1)},\, A_{1,3}^{(-2)}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array}\right]$$

$$\xrightarrow{M_2^{(-1)},\, A_{2,3}^{(1)}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right]$$

- The last row reads $0 = b_3 - b_1 - b_2$, so the system is consistent precisely when $b_3 = b_1 + b_2$.

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}, \qquad b_1, b_2 \text{ arbitrary}$$

EXAMPLE 4

…and a case with no condition at all

$$x_1 + 2x_2 + 3x_3 = b_1$$
$$2x_1 + 5x_2 + 3x_3 = b_2$$
$$x_1 + 8x_3 = b_3$$

$$\rightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -40b_1 + 16b_2 + 9b_3 \\ 0 & 1 & 0 & 13b_1 - 5b_2 - 3b_3 \\ 0 & 0 & 1 & 5b_1 - 2b_2 - b_3 \end{array}\right]$$

- There are NO restrictions on b_1, b_2, b_3: every right-hand side gives exactly one solution.

> Compare the two examples. In Example 3 the coefficient matrix is singular, and a condition on b appears in the last row. In Example 4 it is the invertible matrix of Section 1.5, so no condition can appear — and the answer is literally the rows of A^{-1} acting on b.

SECTION 1.7

Diagonal, Triangular, and Symmetric Matrices

GOAL — Recognise three shapes of matrix whose inverses, powers and products can be written down almost by inspection.

1. Diagonal matrices: inverse and powers
2. Multiplying by a diagonal matrix
3. Triangular matrices and their properties
4. Symmetric matrices
5. Products of symmetric matrices
6. AA^T and A^TA

DEFINITION

Diagonal matrices

A square matrix in which every entry OFF the main diagonal is zero is a DIAGONAL matrix.

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

- D is invertible if and only if every $d_i \neq 0$, and then the inverse is diagonal as well:

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

FORMULA

Powers of a diagonal matrix

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- and the same formula holds for negative k whenever D is invertible.

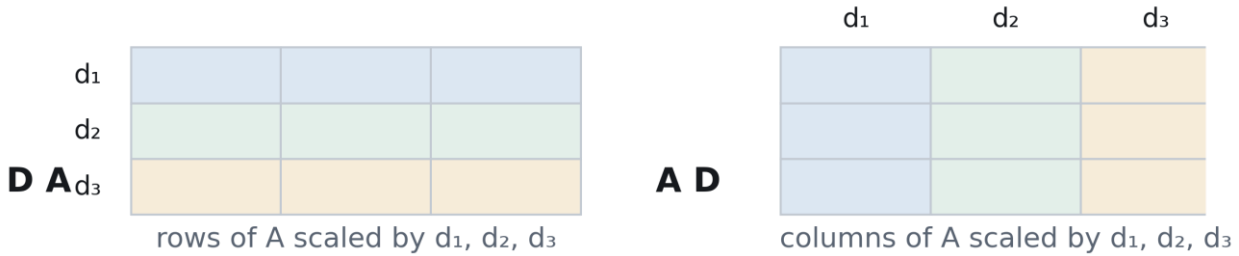
Example 1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}, \quad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/243 & 0 \\ 0 & 0 & 1/32 \end{bmatrix}$$

TECHNIQUE

Multiplying by a diagonal matrix



To form DA, multiply the successive ROWS of A by the successive diagonal entries of D. To form AD, multiply the successive COLUMNS of A by them.

- This is one of the few places where the order of a product has a picture attached to it. Get the picture right and you will never need to compute DA entry by entry again.

DEFINITION

Triangular matrices

- A square matrix with all entries BELOW the main diagonal zero is UPPER TRIANGULAR; with all entries ABOVE it zero, LOWER TRIANGULAR. Either kind is called triangular.

UPPER TRIANGULAR

*	*	*	*
0	*	*	*
0	0	*	*
0	0	0	*

$a_{ij} = 0$ when $i > j$

LOWER TRIANGULAR

*	0	0	0
*	*	0	0
*	*	*	0
*	*	*	*

$a_{ij} = 0$ when $i < j$

- Two useful restatements: A is upper triangular exactly when the i-th ROW starts with at least $i - 1$ zeros; A is lower triangular exactly when the j-th COLUMN starts with at least $j - 1$ zeros.

A diagonal matrix is both. And a SQUARE matrix in row echelon form is upper triangular.

THEOREM 1.7.1

Properties of triangular matrices

(a) The transpose of a lower triangular matrix is upper triangular, and vice versa.

(b) A product of lower triangular matrices is lower triangular; a product of upper triangular matrices is upper triangular.

(c) A triangular matrix is invertible if and only if all of its diagonal entries are nonzero.

(d) The inverse of an invertible lower triangular matrix is lower triangular, and likewise for upper.

> Part (c) is the one you will use most: invertibility of a triangular matrix is decided by GLANCING at the diagonal. Parts (c) and (d) are proved in Chapter 2, once determinants are available.

EXAMPLE 3

Computations with triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

- By part (c), A is invertible (1,2,5 all nonzero) and B is not (a zero on the diagonal). By (b) and (d), A^{-1}, AB and BA must all be upper triangular:

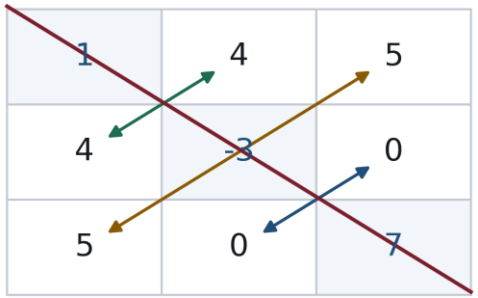
$$A^{-1} = \begin{bmatrix} 1 & -3/2 & 7/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}, \; AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \; BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

- Look at the diagonals of AB and BA: both are $(3,0,5)$, the entrywise products of the diagonals of A and B. That happens for every pair of triangular matrices of the same kind.

DEFINITION 1

Symmetric matrices

A square matrix A is SYMMETRIC if $A = A^T$; equivalently, if $(A)_{ij} = (A)_{ji}$ for all i and j.



mirror images across the main diagonal must match

- The diagonal entries are unrestricted. It is the MIRROR PAIRS that must agree, which is why symmetry is recognised by inspection rather than by computing a transpose.

THEOREM 1.7.2

Symmetry survives sums and scalars

- If A and B are symmetric of the same size and k is a scalar, then

(a) A^T is symmetric. (b) $A + B$ and $A - B$ are symmetric. (c) kA is symmetric.

- Each follows immediately from Theorem 1.4.8: transpose distributes over sums and pulls scalars straight through.

> What is missing from the list is the PRODUCT. That omission is the content of the next theorem.

THEOREM 1.7.3

A product of symmetric matrices — the catch

- For symmetric A and B, Theorem 1.4.8(e) gives $(AB)^T = B^T A^T = BA$. So AB is symmetric exactly when $AB = BA$.

> The product of two symmetric matrices is symmetric IF AND ONLY IF the matrices commute.

Example 5 — one of each

$$\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}-4 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}-2 & 1\\ -5 & 2\end{bmatrix} \qquad \text{not symmetric}$$

$$\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}-4 & 3\\ 3 & -1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix} \qquad \text{symmetric}$$

- So the factors in the first line do not commute, and those in the second do — read off without computing either BA.

THEOREM 1.7.4

The inverse of a symmetric matrix

> If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Proof

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$$

The first equality is Theorem 1.4.9 — transpose and inverse commute. The second uses $A^T = A$. ■

Symmetry alone does not give invertibility: a diagonal matrix with a zero on the diagonal is symmetric and singular. The theorem only says that IF the inverse exists, it inherits the symmetry.

THEOREM 1.7.5

AA^T and A^TA

- For any $m \times n$ matrix A — square or not — both products are symmetric, because

$$\left(AA^T\right)^T = \left(A^T\right)^T A^T = AA^T, \qquad \left(A^TA\right)^T = A^T\left(A^T\right)^T = A^TA$$

- AA^T has size $m \times m$ and A^TA has size $n \times n$. If A is invertible, both are invertible as well, being products of invertible matrices.

Example 6

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}, \quad AA^T = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

SUMMARY

Unit 1C in one slide

1.5 — a row operation is a matrix

- Each elementary row operation is reversible and is realized by LEFT multiplication by an elementary matrix. Every elementary matrix is invertible, and $\mathrm{RREF}([A \mid I]) = \left[I \mid A^{-1}\right]$ both tests invertibility and produces the inverse.

1.6 — invertibility as a statement about systems

- A system has zero, one, or infinitely many solutions. For a square A, invertibility is the same as: only the trivial solution; RREF is I; a product of elementary matrices; $Ax = b$ consistent for every b; $Ax = b$ uniquely solvable for every b. When A is singular, elimination with symbolic b reveals the condition for consistency.

1.7 — three shapes worth recognising

- Diagonal: inverse, powers and products by inspection. Triangular: invertible exactly when the diagonal has no zero, and the shape is inherited by transposes, products and inverses. Symmetric: preserved by $+$, $-$, scalars and inversion, but by products only when the factors commute; AA^T and A^TA are always symmetric.