1.5 Elementary Matrices and Finding A^{-1}

Elementary row operation are reversible:

$$A \xrightarrow{M_i^c} B \xrightarrow{M_i^{\frac{1}{c}}} A, (c \neq 0)$$

$$A \xrightarrow{I_{ij}} B \xrightarrow{I_{ij}} A$$

$$A \xrightarrow{A_{ij}^c} B \xrightarrow{A_{ij}^{-c}} A$$

Definition Row equivalence

Two matrices are called row equivalent if one (hence each) is obtained from the other by a sequence of elementary row operations.

NOTE:

- Row equivalence is an **equivalence relation** on matrices.
- Since RREF is unique, each class contains one and only one such matrix.
- We can use this fact to verify if two given matrices A and B are row equivalent by checking if $RREF(A) = RREF(B)$.

Example: Checking Equivalence of Matrices

Are $\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ row equivalent?

Sol. Since $\mathrm{RREF}\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathrm{RREF}\left(\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are not equal, then A and B are not row equivalent.

Definition Elementary Matrices

A matrix E is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation.

Example Classifying 2×2 Matrices

List all possible elementary matrices of size 2×2.

Sol.

- For any real number $c \neq 0$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{M_1^c} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{M_2^c} \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$.
- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{I_{12}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- For any real number c, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{A_{12}^c} \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{A_{21}^c} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$.

Example Elementary Matrices

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem Row operations by (left) Matrix Multiplication

Let $A \in M_{m\times n}$. If E is an elementary matrix resulted from I_m, then EA is the matrix that results from A using the same row operation.

Example Using Elementary Matrices

$$\begin{bmatrix} 4 & 5 & -6 & 1 & -1 \\ -1 & 2 & -1 & 10 & 3 \\ 3 & 0 & 4 & -4 & 7 \end{bmatrix} \xrightarrow{A_{21}^{4}} \begin{bmatrix} 0 & 13 & -10 & 41 & 11 \\ -1 & 2 & -1 & 10 & 3 \\ 3 & 0 & 4 & -4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 5 & -6 & 1 & -1 \\ -1 & 2 & -1 & 10 & 3 \\ 3 & 0 & 4 & -4 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 13 & -10 & 41 & 11 \\ -1 & 2 & -1 & 10 & 3 \\ 3 & 0 & 4 & -4 & 7 \end{bmatrix}$$

Theorem Elementary Matrices are Invertible

Every elementary matrix E is invertible by an elementary matrix.

Proof. Produce E' using the inverse row operation on I that resulted with E. Using the fact that inverse row operations cancel the effect of each other, it follows that

$$EE' = E'E = I$$

Thus E' is the inverse of E.

Example Inverses of Elementary Matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Theorem Equivalent Statements for a Square Matrix

Let $A \in M_{n\times n}$. The following are equivalent statements.

1. A is invertible.
2. $Ax = 0$ has only the trivial solution.
3. $RREF(A) = I$.
4. A is a product of elementary matrices.

Proof.

Algorithm Finding the Inverse of a Square Matrix A if it exists

$RREF([A|I]) = [I|A^{-1}]$. Otherwise A is not invertible.

Example Finding the Inverse Using Row Operations

Find A^{-1} if it exists. a) $A = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix}$. b) $A = \begin{bmatrix} 3 & 3 & 6 \\ 0 & 1 & 2 \\ -2 & 0 & 0 \end{bmatrix}$

Sol. a) $\left[\begin{array}{cc|cc} -4 & -2 & 1 & 0 \\ 5 & 5 & 0 & 1 \end{array}\right] \xrightarrow{A_{21}^{1}} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 1 \\ 5 & 5 & 0 & 1 \end{array}\right] \xrightarrow{A_{12}^{-5}} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 1 \\ 0 & -10 & -5 & -4 \end{array}\right]$

$\xrightarrow{M_2^{\frac{-1}{5}}} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{2}{5} \end{array}\right] \xrightarrow{A_{21}^{-3}} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & -\frac{1}{5} \\ 0 & 1 & \frac{1}{2} & \frac{2}{5} \end{array}\right]$

Sol. b)

$$\left[\begin{array}{ccc|ccc} 3 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \overset{A_{31}^{1}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{A_{13}^{2}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 6 & 12 & 2 & 0 & 3 \end{array}\right] \overset{A_{23}^{-6}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -6 & 3 \end{array}\right]$$

From the third row we conclude that A is not invertible.

1.6 More on Linear Systems and Invertible Matrices

Theorem Number of Solutions of a Linear System

A system of linear equations can only have zero, one, or infinitely many solutions.

Proof. It suffices to show if the system $Ax = b$ has more than one solution then it has infinitely many solutions. Assume $x_1 \neq x_2$ are solutions, then $x_1 - x_2 \neq 0$ and $x_1 + \lambda(x_1 - x_2)$ for $\lambda \in \mathbb{R}$ is an infinite set. Observe that

$$A(x_1 + \lambda(x_1 - x_2)) = Ax_1 + A\lambda(x_1 - x_2) = Ax_1 + \lambda(Ax_1 - Ax_2) = b + \lambda(b - b) = b$$

So, the system $Ax = b$ has infinitely many solutions.

Solving Systems with a Common Coefficient Matrix $Ax = b_1, Ax = b_2, \dots, Ax = b_k$

- If A is square invertible matrix, then the solutions are

$$x_1 = A^{-1}b_1, x_2 = A^{-1}b_2, \dots, x_k = A^{-1}b_k$$

- In general, we can use Gauss or Gauss-Jordan on the augmented matrix

$$[A|b_1|b_2 \mid \dots |b_k]$$

Theorem Properties of Invertible Matrices

If $A, B \in M_{n\times n}$ satisfy $BA = I$, then $AB = I$.

Proof. It suffices to show A is invertible since then

$$BA = I \Longrightarrow (BA)A^{-1} = IA^{-1} \Rightarrow B\left(AA^{-1}\right) = A^{-1} \Rightarrow BI = A^{-1} \Rightarrow B = A^{-1} \Longrightarrow AB = AA^{-1} = I$$

To show that A is invertible, we will show that the system $Ax = 0$ has only the trivial solution by multiplying both sides by B to get $BAx = B0 \Longrightarrow Ix = 0 \Longrightarrow x = 0$.

Corollary This means we only need to the inverse from one side.

Theorem Equivalent Statements for a Square Matrix

Let $A \in M_{n\times n}$. The following are equivalent statements.

1. A is invertible.
2. $Ax = 0$ has only the trivial solution.
3. $RREF(A) = I$.
4. A is a product of elementary matrices.
5. $Ax = b$ is consistent for every $n \times 1$ matrix b.
6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

Proof. We have proved the equivalence of 1 up to 4. It suffices to prove $1 \Rightarrow 6 \Longrightarrow 5 \Longrightarrow 1$. But we already proved $1 \Rightarrow 6$ and $6 \Rightarrow 5$ is obvious, so it remains to show $5 \Rightarrow 1$. If the system $Ax = b$ is consistent for every b, then we have solutions to the systems: $Ax = e_1, Ax = e_2,\ldots, Ax = e_n$, say $x = x_1, x = x_2, \ldots, x = x_n$. Observe that

$$A[x_1|x_2|\ldots|x_n] = [Ax_1|Ax_2|\ldots|Ax_n] = [e_1|e_2|\ldots|e_n] = I$$

Hence A is invertible.

Theorem Property of Invertible Matrices

Let $A, B \in M_{n\times n}$. If AB is invertible, then both A and B must be invertible.

Proof.

A Fundamental Problem Find all possible b for which the system $Ax = b$ is consistent.

NOTE:

- If A is a square and invertible then $Ax = b$ is consistent for any b.
- If A is not a square or a square but not invertible, then usually there must be some conditions on b so that $Ax = b$ is consistent.

Example Checking for Consistency conditions

Determine the conditions, if any, on b_1, b_2, and b_3 for the following system to be consistent.

$$\begin{aligned} -x - y - z &= b_1 \\ 3x + 2y + 2 &= b_2 \\ x + y &= b_3 \end{aligned}$$

Sol. Upon reducing the augmented matrix we have

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & b_1 \\ 3 & 2 & 2 & b_2 \\ 1 & 1 & 0 & b_3 \end{array}\right] \xrightarrow{I_{13}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & b_3 \\ 3 & 2 & 2 & b_2 \\ -1 & -1 & -1 & b_1 \end{array}\right] \xrightarrow[A_{13}^{1}]{A_{12}^{-3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & b_3 \\ 0 & -1 & 2 & b_2 - 3b_3 \\ 0 & 0 & -1 & b_1 + b_3 \end{array}\right]$$

$$\xrightarrow[M_3^{-1}]{M_2^{-1}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & b_3 \\ 0 & 1 & -2 & 3b_3 - b_2 \\ 0 & 0 & 1 & -b_1 - b_3 \end{array}\right]$$

Clearly the system has a unique and there are no restrictions on b_1, b_2, and b_3.

Example Checking for Consistency conditions

Determine the conditions, if any, on a, b, and c for the following system to be consistent.

$$x + 2y + 2z = a$$
$$3x - 3y - z = b$$
$$4x - y + z = c$$

Sol. Upon reducing the augmented matrix we have the following

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 3 & -3 & -1 & b \\ 4 & -1 & 1 & c \end{array}\right] \xrightarrow[A_{13}^{-4}]{A_{12}^{-3}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & -9 & -7 & b-3a \\ 0 & -9 & -7 & c-4a \end{array}\right] \xrightarrow{A_{23}^{-1}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & -9 & -7 & b-3a \\ 0 & 0 & 0 & c-a-b \end{array}\right]$$

Note the last row gives the equation $0 = c - a - b$ so the system has a solution if and only if the right side is zero.

1.7 Special Matrices

Diagonal Matrices D of size $n \times n$ can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

D is invertible if and only if all of its diagonal entries are nonzero, in which case

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

Powers of Diagonal Matrices D are easily computed

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Example Computing inverses and Powers of Diagonal Matrices

If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{-3} = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ 0 & -\frac{1}{27} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying by a Diagonal Matrix D

- To multiply a matrix A on the left by D, multiply successive rows of A by the successive diagonal entries of D.
- To multiply A on the right by D, multiply successive columns of A by the successive diagonal entries of D.

Theorem Properties of Triangular Matrices

(a) The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.
(b) The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
(c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.
(d) The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Definition Symmetric Matrices

A square matrix A is said to be symmetric if $A^T = A$.

Example Symmetric Matrices

$$\begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 & 0 & 3 \\ -1 & 5 & 4 & 2 \\ 0 & 4 & 6 & -7 \\ 3 & 2 & -7 & 1 \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \quad \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix}$$

Theorem Properties of Symmetric Matrices

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

(a) A^T is symmetric.

(b) $A + B$ is symmetric.

(c) kA is symmetric.

(d) If A is invertible then A^{-1} is symmetric.

Theorem Producing Symmetric Matrices

If A is any matrix, then $A + A^T$ and AA^T are symmetric.