Math 204
Differential Equations
Exact Differential Equations

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Exact Differential Equations

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Example

Solve

$$y\,dx + x\,dy = 0$$

Exact Equations

The total differential of a function $f(x, y)$ is

$$df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

provided that the partial derivatives of the function f is exists.

Exact Equations

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$$df(x, y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

provided that the partial derivatives of the function f is exists.

If $f(x, y) = c$, then $df = 0$.

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Exact Equations

Example

$$x^2 - 5xy + y^3 = c$$

Exact Equations

Example

$$x^2 - 5xy + y^3 = c$$

$$(2x - 5y)\,dx + (3y^2 - 5x)\,dy = 0$$

Exact Equations

Example

$$x^2 - 5xy + y^3 = c$$

$$(2x - 5y)\,dx + (3y^2 - 5x)\,dy = 0$$

$$d(x^2 - 5xy + y^3) = 0$$

Exact Equations

Definition

A differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called exact, if there is a function $f(x, y)$ such that

$$df(x, y) = M(x, y)dx + N(x, y)dy = 0$$

Exact Equations

If

$$M(x,y)dx + N(x,y)dy = 0$$

is exact, then $df = 0$, and the solution is

$$f(x,y) = c$$

Exact Equations

Theorem

If $M, N, \frac{\partial M}{\partial y}$, *and* $\frac{\partial N}{\partial x}$ *are continuous on a region* R *in* $xy-$*plane, then the differential equation*

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example

Solve

$$2xy\,dx + (x^2 - 1)dy = 0$$

Example

Solve

$$(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0.$$

Example

Solve the initial value problem

$$(\cos x \sin x - xy^2)dx + y(1 - x^2)dy = 0$$

$$y(0) = 2$$

Integrating factor

Integrating factor

It is possible to convert a nonexact differential equation into an exact equation by multiplying by a function $\mu(x, y)$ called an integration factor.

Integrating factor

Definition

A function $\mu(x,y)$ called an integration factor for

$$M(x,y)dx + N(x,y)dy = 0$$

if

$$\mu M(x,y)dx + \mu N(x,y)dy = 0$$

is exact, i,e.,

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

on R, where $\mu(x,y) \neq 0$ for all $(x,y) \in R$

Example

Solve

$$(x + y)dx + x \ln x dy = 0$$

using $\mu(x, y) = \frac{1}{x}$ on $(1, \infty)$

Integrating factor

In order to find the integrating factor, we have tow cases

1. $\mu = \mu(x)$, then
$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$
2. $\mu = \mu(y)$, then
$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

Integrating factor

Example

Solve

$$xy\,dx + (2x^2 + 3y^2 - 20)dy = 0$$

where $x \neq 0, y > 0$.

Example

Solve

$$(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$$

where $x(x + 2y) \neq 0$.