PHYSICS 501 – Fall 2019
2nd HOMEWORK
Prof. V. Lempesis
Hand in by Monday 7th October 2019 at 23:59

1. Calculate the quantity $\vec{\nabla}\cdot\mathbf{r}r^{n-1}$ (Hint: See question 2.10).

(5 marks)

Solution:

Remember that in question 2.10 we proved that

$$\vec{\nabla}\cdot\mathbf{r}f(r)=3f(r)+r\frac{df}{dr}$$

In this problem we have that $f(r)=r^{n-1}$ so,

$$\vec{\nabla}\cdot\mathbf{r}f(r)=3f(r)+r\frac{df}{dr}=3r^{n-1}+r\frac{dr^{n-1}}{dr}=3r^{n-1}+r(n-1)r^{n-2}$$

$$=3r^{n-1}+(n-1)r^{n-1}=(n+2)r^{n-1}$$

2. Show that if a vector **A** is irrotational, then $\mathbf{A}\times\mathbf{r}$ is solenoidal.

(5 marks)

Solution:

$$\vec{\nabla}\left(\mathbf{A}\times\mathbf{r}\right)=\mathbf{r}\cdot\left(\vec{\nabla}\times\mathbf{A}\right)-\mathbf{A}\cdot\left(\vec{\nabla}\times\mathbf{r}\right)$$

But if the vector **A** is irrotational then $\vec{\nabla}\times\mathbf{A}=0$. Thus

$$\vec{\nabla}\left(\mathbf{A}\times\mathbf{r}\right)=-\mathbf{A}\cdot\left(\vec{\nabla}\times\mathbf{r}\right).$$

But

$$\vec{\nabla}\times\mathbf{r}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x & y & z\end{vmatrix}=$$

$$\mathbf{i}\left(\partial z/\partial y-\partial y/\partial z\right)-\mathbf{j}\left(\partial z/\partial x-\partial x/\partial z\right)+\mathbf{k}\left(\partial y/\partial x-\partial x/\partial y\right)=0$$

Thus $\vec{\nabla}\left(\mathbf{A}\times\mathbf{r}\right)=0$ so the vector $\mathbf{A}\times\mathbf{r}$ is solenoidal.

Vasileios Lembessis 7/10/2019 19:10
Comment [1]: You must show this in your answer.

3. Classically the angular momentum is defined by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p}$ is the momentum. To go from classical mechanics to quantum mechanics we replace $\mathbf{p}$ with the operator $-i\hbar\vec{\nabla}$. Find the Cartesian components of the angular momentum operator.

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(5 marks)

Solution

$$\mathbf{p} = -i\hbar\vec{\nabla} \Rightarrow \mathbf{p} = -i\hbar\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = -i\hbar\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} = -i\hbar\left[\mathbf{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) - \mathbf{j}\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) + \mathbf{k}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]$$

Thus

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \quad L_y = i\hbar\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right), \quad L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

4. Show the relation $\mathbf{L} \times \mathbf{L} = i\hbar\mathbf{L}$ for the quantum mechanical angular momentum operator $\mathbf{L}$. (Hint: Show first that $\left(\mathbf{L} \times \mathbf{L}\right)_x = i\hbar\mathbf{L}_x$, i.e. work for the x-component, the other components go similarly).

(5 marks)

$$\mathbf{L} \times \mathbf{L} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ L_x & L_y & L_z \\ L_x & L_y & L_z \end{vmatrix} = \left[\mathbf{i}\left(L_yL_z - L_zL_x\right) - \mathbf{j}\left(L_xL_z - L_xL_z\right) + \mathbf{k}\left(L_xL_y - L_yL_x\right)\right]$$

Thus for the x-component $\left(\mathbf{L} \times \mathbf{L}\right)_x$ of the cross product we have:

$$\left(\mathbf{L} \times \mathbf{L}\right)_x = \left(L_yL_z - L_zL_x\right) = -\hbar^2\left\{\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\}$$

$$= -\hbar^2 \left\{ z\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) - z\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial x}\right) - x\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\right) + x\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial x}\right) \right.$$

$$\left. - x\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}\right) + x\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial z}\right) + y\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial z}\right) \right\}$$

$$= -\hbar^2 \left\{ z\frac{\partial}{\partial y} + zx\frac{\partial^2}{\partial x \partial y} - zy\frac{\partial^2}{\partial x^2} - x^2\frac{\partial^2}{\partial z \partial x} + xy\frac{\partial^2}{\partial z \partial x} - xz\frac{\partial^2}{\partial y \partial x} + x^2\frac{\partial^2}{\partial z \partial y} + yz\frac{\partial^2}{\partial x^2} - y\frac{\partial}{\partial z} - yx\frac{\partial^2}{\partial z \partial x} \right\}$$

$$= -\hbar^2 \left\{ z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \right\} = i^2\hbar^2 \left\{ z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \right\} = i\hbar \left\{ i\hbar \left[z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \right] \right\} = i\hbar L_x$$

Similarly for the other two components

Please send your answers in pdf form (typed or in clearly handwritten form) in my email address (vlempesis@ksu.edu.sa). Please use ONE file for your entire homework NOT one file per page. Please do not forget to put your name and your ID number on it)
Your deadline is on Monday 7th October 2019 at 23:59.

Vasileios Lembessis 8/10/2019 10:29
Comment [2]: Some of you used the relations [Ly. Lz] =i*hbar*Lz. But you must prove this is true. You cannot use it directly.

Some of you considered that $(L\times L)_x = L_x \times (r_x \times p_x)$ which is completely wrong