Q4: Let $f(x) = \sqrt{x - x^2}$ and $p_2(x)$ be the quadratic Lagrange interpolating polynomial which interpolates f at $x_0 = 0, x_1 = \alpha$ and $x_2 = 1$. Find the largest value of α, in the interval $(0, 1)$, for which

$$f(0.5) - p_2(0.5) = -0.25.$$

Solution. Consider the quadratic Lagrange interpolating polynomial as follows:

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2).$$

At the given values of $x_0 = 0, x_1 = \alpha, x_2 = 1$, we have, $f(0) = 0, f(\alpha) = \sqrt{\alpha - \alpha^2}$ and $f(1) = 0$, gives

$$p_2(x) = L_0(x)(0) + L_1(x)(\sqrt{\alpha - \alpha^2}) + L_2(x)(0),$$

where

$$L_1(x) = \frac{(x-0)(x-1)}{(\alpha-0)(\alpha-1)} = \frac{x^2 - x}{\alpha^2 - \alpha}.$$

Thus

$$p_2(x) = \frac{x^2 - x}{\alpha^2 - \alpha}\sqrt{\alpha - \alpha^2} \quad \text{and} \quad p_2(0.5) = \frac{-0.25}{\alpha^2 - \alpha}\sqrt{\alpha - \alpha^2} = \frac{0.25}{\sqrt{\alpha - \alpha^2}}.$$

Given

$$f(0.5) - p_2(0.5) = -0.25, \quad \text{gives} \quad p_2(0.5) = f(0.5) + 0.25 = 0.5 + 0.25 = 0.75,$$

so

$$\frac{0.25}{\sqrt{\alpha - \alpha^2}} = 0.75, \quad \text{or} \quad \sqrt{\alpha - \alpha^2} = \frac{1}{3}.$$

Thus, taking square on both sides, we get

$$\alpha - \alpha^2 = \frac{1}{9}, \quad \text{or} \qquad 9\alpha^2 - 9\alpha + 1 = 0.$$

Solving this equation for α, we get, $\alpha = 0.127322$ or $\alpha = 0.872678$. Thus $\alpha = 0.872678$, the required largest value of α in the given interval $(0, 1)$. •