King Saud University: Mathematics Department Math-254
Second Semester 1447 H Second Midterm Exam.
Maximum Marks = 25 Time: 90 mins.

Questions: (6 + 7 + 6 + 6) Marks

Question 1:

Use LU decomposition by **Dollittle's method** to find the value(s) of nonzero α for which the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix},$$

is **consistent**(only unique solution). Solve the **consistent system**(unique solution).

Question 2:

Consider the following linear system of equations

$$\begin{array}{rcrcrcr} 2x_1 & + & \alpha x_2 & & & = & 3 \\ \alpha x_1 & + & 8x_2 & + & \alpha x_3 & = & 10 \\ & & \alpha x_2 & + & 2x_3 & = & 3 \end{array}$$

Find an **interval** (a, b) for α such that Jacobi iterative method will converge using $\|T_J\|_\infty < 1$. **Compute** an error bound $\|\mathbf{x} - \mathbf{x}^{(10)}\|$ using the largest positive integer value of α and the initial solution $\mathbf{x}^{(0)} = [0.5, 0.5, 0.5]^T$. How many **iterations** needed to get an accuracy within 10^{-4}.

Question 3:

Let $f(x) = \sqrt{2x+1}$, and

$$x_0 = 1.7,\ x_1 = 1.8,\ x_2 = 1.9,\ x_3 = 2.1,\ x_4 = 2.3,\ x_5 = 2.4,\ x_6 = 3.1,\ x_7 = 4.2.$$

Use the quadratic Lagrange interpolating polynomial by selecting the best points to **estimate** $\sqrt{5}$. Also, **compute** the error bound and the absolute error.

Question 4:

Determine an appropriate **step size** h (2 decimal places) to construct a table of the function $f(x) = x + \ln(x+1)$ on the interval $[2, 3]$ when the error for **linear Lagrange interpolation** is to be bounded by 8.9×10^{-4}. Then use the constructed table to find the approximation of $f(2.6)$ using **linear Lagrange polynomial** and **compute** the absolute error.

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Question 1:

Use LU decomposition by **Dollittle's method** to find the value(s) of nonzero α for which the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix},$$

is **consistent**(only unique solution). Solve the **consistent system**(unique solution).

Solution. Since we know that

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = LU.$$

Using $m_{21} = \frac{2\alpha}{\alpha} = 2 = l_{21}$, $m_{31} = \frac{1}{\alpha} = l_{31}$, and $m_{32} = \dfrac{(3\alpha - 4)}{(-9\alpha)} = l_{32}(\alpha \neq 0)$, gives

$$\begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & \frac{(3\alpha-4)}{\alpha} & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix} \equiv \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix}.$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Thus

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{\alpha} & \frac{(3\alpha-4)}{(-9\alpha)} & 1 \end{pmatrix} \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix},$$

which is the required decomposition of A. If $\det(A) \neq 0$,then the given linear system has unique solution, that is, if

$$det(A) = det(U) = \frac{-9\alpha(\alpha^2 - 1)}{\alpha} = -9(\alpha^2 - 1) = (\alpha^2 - 1) \neq 0,$$

which gives, $\alpha \neq -1$ or $\alpha \neq 1$.
If $\det(A) \neq 0$, then we can find the unique solution for system for nonzero α $(\neq \pm 1)$ as follows:

$$L\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{\alpha} & \frac{(3\alpha-4)}{(-9\alpha)} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix} = \mathbf{b}.$$

Performing forward substitution yields

$$\begin{array}{lllllllll} y_1 & & & & & = 6, & \text{gives } y_1 & = & 6, \\ 2y_1 & + & y_2 & & & = 3, & \text{gives } y_2 & = & 3 - 12, \\ \frac{1}{\alpha}y_1 & + & \frac{(3\alpha-4)}{(-9\alpha)}y_2 & + & y_3 & = 5, & \text{gives } y_3 & = & 5 - \frac{(2+3\alpha)}{\alpha}. \end{array}$$

Thus it yields,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ \frac{2(\alpha-1)}{\alpha} \end{pmatrix}.$$

Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ \frac{2\alpha-2}{\alpha} \end{pmatrix},$$

performing backward substitution yields

$$\begin{array}{rcrcrcrlrcr} \alpha x_1 & + & 4x_2 & + & x_3 & = & 6, & \text{gives} & x_1 & = & \frac{1}{\alpha}(2 - \frac{2}{\alpha+1}), \\ & - & 9x_2 & & & = & -9, & \text{gives} & x_2 & = & 1, \\ & & & & (\frac{\alpha^2-1}{\alpha})x_3 & = & \frac{2(\alpha-1)}{\alpha}, & \text{gives} & x_3 & = & (\frac{2(\alpha-1)}{\alpha})(\frac{\alpha}{(\alpha^2-1)}), \end{array}$$

and it yields,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{\alpha+1} \\ 1 \\ \frac{2}{\alpha+1} \end{pmatrix},$$

the required unique solution for any α except $\alpha \neq \pm 1$. •

Question 2:
Consider the following linear system of equations

$$\begin{array}{rcrcrcr} 2x_1 & + & \alpha x_2 & & & = & 3 \\ \alpha x_1 & + & 8x_2 & + & \alpha x_3 & = & 10 \\ & & \alpha x_2 & + & 2x_3 & = & 3 \end{array}$$

Find an **interval** (a,b) for α such that Jacobi iterative method will converge using $\|T_J\|_\infty < 1$. **Compute** the error bound $\|\mathbf{x} - \mathbf{x}^{(10)}\|$ using the largest positive integer value of α and the initial solution $\mathbf{x}^{(0)} = [0.5, 0.5, 0.5]^T$. How many **iterations** needed to get an accuracy within 10^{-4}.

Solution. Since we know

$$A = \begin{pmatrix} 2 & \alpha & 0 \\ \alpha & 8 & \alpha \\ 0 & \alpha & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix} = L + U + D.$$

Since the Jacobi iteration matrix is defined as

$$T_J = -D^{-1}(L+U) = -\begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix} = -\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix},$$

and then multiplying the matrices, we have

$$T_J = \begin{pmatrix} 0 & -\alpha/2 & 0 \\ -\alpha/8 & 0 & -\alpha/8 \\ 0 & -\alpha/2 & 0 \end{pmatrix}.$$

Then the l_∞ norm of the matrix T_J is

$$\|T_J\|_\infty = \max\left\{ \frac{|\alpha|}{2}, \frac{|\alpha|}{4}, \frac{|\alpha|}{2} \right\} = \frac{|\alpha|}{2} < 1.$$

Thus the Jacobi method will converge for the given linear system for $-2 < \alpha < 2$, that is, any value of α in the interval $(-2, 2)$.

The Jacobi method for the given system using the largest positive integer value of α which is, $\alpha = 1$, gives

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{2}\Big[3 - x_2^{(k)}\Big] \\ x_2^{(k+1)} &= \frac{1}{8}\Big[10 - x_1^{(k)} - x_3^{(k)}\Big] \\ x_3^{(k+1)} &= \frac{1}{2}\Big[3 - x_2^{(k)}\Big] \end{aligned}$$

Starting with initial approximation $x_1^{(0)} = 0.5, x_2^{(0)} = 0.5, x_3^{(0)} = 0.5$, and for $k = 0$, we obtain the first approximation as

$$\begin{aligned} x_1^{(1)} &= \frac{1}{2}\Big[3 - x_2^{(0)}\Big] &&= \frac{1}{2}\Big[3 - 0.5\Big] = 1.25, \\ x_2^{(1)} &= \frac{1}{8}\Big[10 - x_1^{(0)} - x_3^{(0)}\Big] &&= \frac{1}{8}\Big[10 - 0.5 - 0.5\Big] = 1.125, \\ x_3^{(1)} &= \frac{1}{2}\Big[3 - x_2^{(0)}\Big] &&= \frac{1}{2}\Big[3 - 0.5\Big] = 1.25, \end{aligned}$$

gives,

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1.25 \\ 1.125 \\ 1.25 \end{pmatrix}.$$

Using the error bound formula,

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T_J\|^k}{1 - \|T_J\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|,$$

for $k = 10$, we obtain

$$\|\mathbf{x} - \mathbf{x}^{(10)}\| \leq \frac{(1/2)^{10}}{1 - 1/2} \left\| \begin{pmatrix} 1.25 \\ 1.125 \\ 1.25 \end{pmatrix} - \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} \right\|.$$

Since

$$\left\| \begin{pmatrix} 1.25 \\ 1.125 \\ 1.25 \end{pmatrix} - \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0.75 \\ 0.625 \\ 0.75 \end{pmatrix} \right\| = 0.75,$$

therefore,

$$\|\mathbf{x} - \mathbf{x}^{(10)}\| \leq \left(\frac{1}{2}\right)^9 (0.75) = 0.0015,$$

the required error bound.
To find the number of iterations, we use the error bound formula as

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T_J\|^k}{1 - \|T_J\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \leq 10^{-4}.$$

It gives

$$\frac{(1/2)^k}{1/2}(0.75) \leq 10^{-4}, \quad \text{or} \quad (1/2)^k \leq \frac{10^{-4}}{1.5}.$$

Taking ln on both sides, we obtain,

$$k \ln(1/2) \leq \ln\left(\frac{10^{-4}}{1.5}\right),$$

and solving for k, we get

$$k \geq 13.8727, \qquad \text{or} \qquad k = 14,$$

the required number of iterations. •

Question 3:
Let $f(x) = \sqrt{2x+1}$, and

$$x_0 = 1.7,\ x_1 = 1.8,\ x_2 = 1.9,\ x_3 = 2.1,\ x_4 = 2.3,\ x_5 = 2.4,\ x_6 = 3.1,\ x_7 = 4.2.$$

Use the quadratic Lagrange interpolating polynomial by selecting the best points to **estimate** $\sqrt{5}$. Also, **compute** the error bound and the absolute error.

Solution. Since $f(x) = \sqrt{2x+1}$, to estimate $\sqrt{5}$, we note that $\sqrt{5} = f(2)$. We choose the three best points nearest to $x = 2$, namely

$$x_0 = 1.8, \qquad x_1 = 1.9, \qquad x_2 = 2.1.$$

Then

$$f(1.8) = \sqrt{4.6} \approx 2.1448,$$
$$f(1.9) = \sqrt{4.8} \approx 2.1909,$$
$$f(2.1) = \sqrt{5.2} \approx 2.2804.$$

So the interpolation points are

$$(1.8,\ 2.1448), \qquad (1.9,\ 2.1909), \qquad (2.1,\ 2.2804).$$

The quadratic Lagrange interpolating polynomial is

$$P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x),$$

where

$$L_0(x) = \frac{(x-1.9)(x-2.1)}{(1.8-1.9)(1.8-2.1)},$$
$$L_1(x) = \frac{(x-1.8)(x-2.1)}{(1.9-1.8)(1.9-2.1)},$$
$$L_2(x) = \frac{(x-1.8)(x-1.9)}{(2.1-1.8)(2.1-1.9)}.$$

To estimate $\sqrt{5}$, evaluate at $x = 2$:

$$L_0(2) = \frac{(2-1.9)(2-2.1)}{(1.8-1.9)(1.8-2.1)} = \frac{(0.1)(-0.1)}{(-0.1)(-0.3)} = -\frac{1}{3},$$

$$L_1(2) = \frac{(2-1.8)(2-2.1)}{(1.9-1.8)(1.9-2.1)} = \frac{(0.2)(-0.1)}{(0.1)(-0.2)} = 1,$$

$$L_2(2) = \frac{(2-1.8)(2-1.9)}{(2.1-1.8)(2.1-1.9)} = \frac{(0.2)(0.1)}{(0.3)(0.2)} = \frac{1}{3}.$$

Hence

$$P_2(2) = f(1.8)\left(-\frac{1}{3}\right) + f(1.9)(1) + f(2.1)\left(\frac{1}{3}\right).$$

Substituting the values,

$$P_2(2) = -\frac{1}{3}(2.1448) + 2.1909 + \frac{1}{3}(2.2804) = 2.2361.$$

Therefore,

$$\sqrt{5} = f(2) \approx P_2(2) = 2.2361.$$

The true value is

$$\sqrt{5} \approx 2.2361.$$

So the absolute error is

$$\left|\sqrt{5} - P_2(2)\right| = |2.2361 - 2.2361| = 0, \quad 4\ dp.$$

Now we compute the error bound. For quadratic interpolation,

$$R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x - x_0)(x - x_1)(x - x_2)$$

for some $\xi \in [1.8, 2.1]$.

Since

$$f(x) = (2x + 1)^{1/2},$$

we have

$$\begin{aligned} f'(x) &= (2x + 1)^{-1/2}, \\ f''(x) &= -(2x + 1)^{-3/2}, \\ f^{(3)}(x) &= 3(2x + 1)^{-5/2}. \end{aligned}$$

Therefore,

$$|f^{(3)}(x)| = \frac{3}{(2x + 1)^{5/2}}.$$

On the interval $[1.8, 2.1]$, this is largest at $x = 1.8$. Thus

$$M = \max_{[1.8,2.1]} |f^{(3)}(x)| = \frac{3}{(4.6)^{5/2}} \approx 0.0661.$$

Also,

$$|(2 - 1.8)(2 - 1.9)(2 - 2.1)| = |(0.2)(0.1)(-0.1)| = 0.002.$$

Hence

$$|R_2(2)| \leq \frac{M}{6}(0.002) = \frac{0.0661}{6}(0.002) \approx 2.2035 \times 10^{-5},$$

the required error bound. •

Question 4:

Determine an appropriate **step size** h (2 decimal places) to construct a table of the function $f(x) = x + \ln(x+1)$ on the interval $[2, 3]$ when the error for **linear Lagrange interpolation for equally spaced data points** is to be bounded by 8.9×10^{-4}. Then use the constructed table to find the approximation of $f(2.6)$ using **linear Lagrange polynomial** and **compute** the absolute error.

Solution. Since we know that error bound formula for the linear Lagrange polynomial is

$$|E| \leq \frac{Mh^2}{8} \leq 8.9 \times 10^{-4}, \quad h = x_1 - x_0,$$

$$\frac{Mh^2}{8} \leq 8.9 \times 10^{-4}, \quad \text{where} \quad M = \max_{2 \leq x \leq 3} |f''(x)|.$$

Since

$$f'(x) = 1 + \frac{1}{x+1} \quad \text{and} \quad f''(x) = -\frac{1}{(x+1)^2}, \quad M = \max_{2 \leq x \leq 3} \left| \frac{-1}{(x+1)^2} \right| = \frac{1}{9}.$$

Thus

$$h^2 \leq (8)(8.9)(9) \times 10^{-4} = 0.0641, \quad \text{gives} \quad h \leq 0.2531 \approx 0.25.$$

Hence we obtain the required table

x	2	2.25	2.5	2.75	3
$f(x)$	3.0986	3.4287	3.7528	4.0718	4.3863

for the given function $f(x)$ on $[2, 3]$ with step size $h = 0.25$.

As $x = 2.6$, so take two points $x_0 = 2.5$ and $x_1 = 2.75$, then linear Lagrange polynomial $p_1(x)$

$$f(x) = p_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1),$$

and interpolating $f(2.6)$ at these points is

$$f(2.6) \approx p_1(2.6) = \frac{(2.6 - 2.75)}{(2.5 - 2.75)}(3.7528) + \frac{(2.6 - 2.5)}{(2.75 - 2.5)}(4.0718) = 3.8804.$$

To compute absolute error in the linear interpolation of $f(x)$, we do as follows

$$|f(2.6) - p_1(2.6)| = |(2.6 + \ln(3.6)) - 3.8804| = |3.8809 - 3.8804| = 0.0005,$$

the required absolute error. •