King Saud University: Mathematics Department Math-254
First Semester 1445 H Second Midterm Exam.
Maximum Marks = 25 Time: 90 mins.

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**Question 1:** Use LU-factorization method with Doolittle's method ($l_{ii} = 1$) to find the solution of the consistent system for $\alpha \neq 3$. [7 Marks]

$$\begin{aligned} x_1 + x_2 &= 1 \\ 3x_1 + \alpha x_2 + 5x_3 &= 8 \\ 7x_2 + 3x_3 &= 3 \end{aligned}$$

**Question 2:** Consider the following linear system of equations [6 Marks]

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ x_1 + 8x_2 + x_3 &= 10 \\ x_2 + 2x_3 &= 3 \end{aligned}$$

If $\mathbf{x} = [1, 1, 1]^T$ be the exact solution of the system, then using Jacobi iterative method and $\mathbf{x}^{(0)} = [0.5, 0.5, 0.5]^T$, compute the absolute error $\|\mathbf{x} - \mathbf{x}^{(2)}\|$. How many iterations needed to get an accuracy within $10^{-4}$ using Jacobi iterative method.

**Question 3:** Consider a linear system $A\mathbf{x} = \mathbf{b}$, where [6 Marks]

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

If $\mathbf{b}$ is changed to $\mathbf{b}^* = [1, 1, 1.99]^T$, then use the residual vector $\mathbf{r}$ to find the relative error in the solution to the linear system $A\mathbf{x} = \mathbf{b}$.

**Question 4:** Use the following table to find the best approximation of $f(0.6)$ by using quadratic Lagrange interpolating polynomial for equally spaced data points [6 Marks]

| $x$ | 0.15 | 0.2 | 0.3 | 0.5 | 0.55 | 0.8 | 1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | −0.0427 | −0.0644 | −0.1084 | −0.1733 | −0.1808 | −0.1428 | 0 |

The above table is for $f(x) = x^2 \ln x$. Determine the number of points when the error for quadratic Lagrange interpolation for equally spaced data points is to be bounded by $10^{-6}$.

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**Question 1:** Use LU-factorization method with Doolittle's method ($l_{ii} = 1$) to find the solution of the consistent system for $\alpha \neq 3$. [7 Marks]

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & & & = & 1 \\ 3x_1 & + & \alpha x_2 & + & 5x_3 & = & 8 \\ & & 7x_2 & + & 3x_3 & = & 3 \end{array}$$

**Solution.** Using Simple Gauss-elimination method, we can easily find factorization of $A$ as

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & \alpha & 5 \\ 0 & 7 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \frac{7}{(\alpha-3)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & (\alpha-3) & 5 \\ 0 & 0 & \frac{(3\alpha-44)}{(\alpha-3)} \end{pmatrix} = LU.$$

Since by one of the property of the determinant

$$\det(A) = \det(LU) = \det(L)\det(U).$$

So when using LU decomposition by Doolittle's method, then

$$\det(A) = \det(U) = \prod_{i=1}^{n} u_{ii} = (u_{11}u_{22}\cdots u_{nn}),$$

where $\det(L) = 1$ because $L$ is lower-triangular matrix and all its diagonal elements are unity. Thus the determinant of the given matrix $A$ is

$$|A| = |U| = (\alpha-3)\frac{(3\alpha-44)}{(\alpha-3)} = (3\alpha-44), \quad \alpha \neq 3.$$

**Many Solutions:**

For many solutions $A$ must be singular, that is, $|A| = 0$,

$$|A| = 3\alpha - 44 = 0, \quad \text{gives} \quad \alpha = 44/3.$$

For this value of $\alpha$ we have the many solutions. By solving the lower-triangular system of the form $L\mathbf{y} = [1, 8, 3]^T$ of the form

$$L\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \frac{3}{5} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix} = \mathbf{b},$$

we obtained the solution $\mathbf{y} = [1, 5, 0]^T$. Now solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ of the form

$$U\mathbf{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{35}{3} & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} = \mathbf{y}.$$

If we choose $x_3 = t \in R$, $t \neq 0$, then, $x_2 = 3(1-t)/7$ and $x_1 = (4+3t)/7$, then the many solutions of the given system is $\mathbf{x}^* = [(4+3t)/7, 3(1-t)/7, t]^T$.

**Unique Solution:**

For unique solution $A$ must be nonsingular, that is, $|A| \neq 0$,

$$|A| = 3\alpha - 44 \neq 0, \quad \text{so} \quad \alpha \in \mathbf{R}, \quad \text{but} \quad \alpha \neq 44/3.$$

By solving the lower-triangular system of the form $L\mathbf{y} = [1, 8, 3]^T$ of the form

$$L\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \frac{7}{(\alpha-3)} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix} = \mathbf{b},$$

we obtained the solution $\mathbf{y} = [1, 5, \frac{(3\alpha-44)}{(\alpha-3)}]^T$. Now solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ of the form

$$U\mathbf{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & (\alpha-3) & 5 \\ 0 & 0 & \frac{(3\alpha-44)}{(\alpha-3)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ \frac{(3\alpha-44)}{(\alpha-3)} \end{pmatrix} = \mathbf{y},$$

and using backward substitution, gives

$$x_1 = 1, \quad x_2 = 0, \quad \text{and} \quad x_3 = 1,$$

the unique solution of the system. •

**Question 2:** Consider the following linear system of equations [6 Marks]

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & & & = & 3 \\ x_1 & + & 8x_2 & + & x_3 & = & 10 \\ & & x_2 & + & 2x_3 & = & 3 \end{array}$$

If $\mathbf{x} = [1, 1, 1]^T$ be the exact solution of the system, then using Jacobi iterative method and $\mathbf{x}^{(0)} = [0.5, 0.5, 0.5]^T$, compute the absolute error $\|\mathbf{x} - \mathbf{x}^{(2)}\|$. How many iterations needed to get an accuracy within $10^{-4}$ using Jacobi iterative method.

**Solution.** As we have,

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 8 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix} = L + U + D.$$

Since the Jacobi iteration matrix is defined as

$$T_J = -D^{-1}(L+U),$$

and by using the given information, we have

$$T_J = \begin{pmatrix} 0 & -1/2 & 0 \\ -1/8 & 0 & -1/8 \\ 0 & -1/2 & 0 \end{pmatrix}.$$

Then the $l_\infty$ norm of the matrix $T_J$ is

$$\|T_J\|_\infty = \max\left\{|-\frac{1}{2}|, |-\frac{1}{4}|, |\frac{1}{2}\right\} = \frac{1}{2} < 1.$$

Thus the Jacobi method will converge for the given linear system.

*(b)* The Jacobi method for the given system is

$$\begin{aligned}
x_1^{(k+1)} &= \frac{1}{2}\Big[3 - x_2^{(k)}\Big] \\
x_2^{(k+1)} &= \frac{1}{8}\Big[10 - x_1^{(k)} - x_3^{(k)}\Big] \\
x_3^{(k+1)} &= \frac{1}{2}\Big[3 - x_2^{(k)}\Big]
\end{aligned}$$

Starting with initial approximation $x_1^{(0)} = 0.5, x_2^{(0)} = 0.5, x_3^{(0)} = 0.5$, and for $k = 0, 1$, we obtain the first and the second approximations as

$$\mathbf{x}^{(1)} = [1.25, 1.125, 1.25]^T \quad \text{and} \quad \mathbf{x}^{(2)} = [0.9375, 0.9375, 0.9375]^T.$$

Thus

$$\|\mathbf{x} - \mathbf{x}^{(2)}\| = \|[0.0625, 0.0625, 0.0625]^T\| = 0.0625,$$

is the absolute error. To find the number of iterations, we use the error bound formula as

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T_J\|^k}{1 - \|T_J\|}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \leq 10^{-4}.$$

It gives

$$\frac{(1/2)^k}{1/2}(0.75) \leq 10^{-4}, \quad \text{or} \quad (1/2)^k \leq \frac{10^{-4}}{1.5}.$$

Taking logarithmic on both sides, we obtain, $k \geq 13.8727, \quad$ or $\quad k = 14$. •

**Question 3:** Consider a linear system $A\mathbf{x} = \mathbf{b}$, where [6 Marks]

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

If $\mathbf{b}$ is changed to $\mathbf{b}^* = [1, 1, 1.99]^T$, then use the residual vector $\mathbf{r}$ to find the relative change in the solution to the linear system $A\mathbf{x} = \mathbf{b}$.

**Solution.** Since the matrix $A$ and its inverse is

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 4/3 & 1 & -8/3 \\ -1/3 & 0 & 2/3 \\ -2/3 & -1 & 7/3 \end{pmatrix}.$$

Then

$$\|A\|_\infty = 5, \|A^{-1}\|_\infty = 5, \qquad K(A) = \|A\|_\infty \|A^{-1}\|_\infty = (5)(5) = 25.$$

Since

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = \mathbf{b} - \mathbf{b}^* = [0, 0, 0.01]^T,$$

and the $l_\infty$-norm of residual vector, $\|\mathbf{r}\|_\infty = 0.01$. From the equation relative error, we get

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{25(0.01)}{2} = 0.1250,$$

the possible relative error in the solution to the given linear system. •

**Question 4:** Use the following table to find the best approximation of $f(0.6)$ by using quadratic Lagrange interpolating polynomial for equally spaced data points [6 Marks]

| $x$ | 0.15 | 0.2 | 0.3 | 0.5 | 0.55 | 0.8 | 1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-0.0427$ | $-0.0644$ | $-0.1084$ | $-0.1733$ | $-0.1808$ | $-0.1428$ | 0 |

The above table is for $f(x) = x^2 \ln x$. Determine the number of points when the error for quadratic Lagrange interpolation for equally spaced data points is to be bounded by $10^{-6}$.

**Solution.** Since given $x = 0.6$, therefore, the best points for the equally spaced data points Lagrange quadratic polynomial are, $x_0 = 0.3, x_1 = 0.55$ and $x_2 = 0.8$ with $h = 0.25$. Consider the quadratic Lagrange interpolating polynomial as

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{1}$$

$$f(0.6) \approx p_2(0.6) = L_0(0.6)(-0.1084) + L_1(0.6)(-0.1808) + L_2(0.6)(-0.1428). \tag{2}$$

The Lagrange coefficients can be calculate as follows:

$$\begin{aligned} L_0(0.6) &= \frac{(0.6-0.55)(0.6-0.8)}{(0.3-0.55)(0.3-0.8)} = -2/25 = -0.08, \\ L_1(0.6) &= \frac{(0.6-0.3)(0.6-0.8)}{(0.55-0.3)(0.55-0.8)} = 24/25 = 0.96, \\ L_2(0.6) &= \frac{(0.6-0.3)(0.6-0.55)}{(0.8-0.3)(0.8-0.55)} = 3/25 = 0.12. \end{aligned}$$

Putting these values of the Lagrange coefficients in (2), we have

$$f(0.4) \approx p_2(0.4) = (-2/25)(-0.1084) + (24/25)(-0.1808) + (3/25)(-0.1428) = -0.1820,$$

which is the required approximation of the given exact solution $0.36 \ln 0.6 \approx -0.1839$.

To compute number of points we have to use error bound for the quadratic Lagrange interpolation for equally spaced data points as

$$|f(x) - p_2(x)| \leq \frac{Mh^3}{9\sqrt{3}} \leq 10^{-6}.$$

As $h = \dfrac{b-a}{n} = \dfrac{0.8-0.3}{n} = \dfrac{0.5}{n}$, so

$$\frac{M(0.5)^3}{(9\sqrt{3})n^3} \leq 10^{-6}.$$

Since

$$M = \max_{0.3 \le x \le 0.8} |f^{(3)}(x)|,$$

and the first three derivatives are

$$f'(x) = 2x \ln x + x, \qquad f''(x) = 2 \ln x + 3, \qquad f^{(3)}(x) = \frac{2}{x},$$

so

$$M = \max_{0.3 \le x \le 0.8} \left| \frac{2}{x} \right| = 20/3 = 6.6667.$$

Hence

$$n^3 \ge \frac{(6.6667)(0.5)^3(10^6)}{9\sqrt{3}},$$

or

$$n \ge \left( \frac{(6.6667)(0.5)^3(10^6)}{9\sqrt{3}} \right)^{1/3},$$

which gives

$$n \ge 37.6709, \qquad \text{gives} \qquad n = 38,$$

and so the number of points is, $n + 1 = 38 + 1 = 39$. •