King Saud University:	**Mathematics Department**	**Math-254**
Second Semester	**1447-48 H**	**First Midterm Examination**
Time: 90 Mins.		**Marks: 25**

Question 1: Show that the following iterative sequences are the fixed point forms to the cubic root of a positive number N

$$x_{n+1} = g_1(x_n) = N x_n^{-2}; \qquad n = 0, 1, \ldots,$$

$$x_{n+1} = g_2(x_n) = \sqrt{\frac{N}{x_n}}; \qquad n = 0, 1, \ldots.$$

Use the convergent sequence to find the third approximation of the cubic root of 8 taking $x_0 = 1.5$ and then compute the absolute error. (6 Marks)

Question 2: Show that the Newton's iterative formula for evaluating 4^{th} root of a positive number N is

$$x_{n+1} = \frac{3}{4} x_n + \frac{N}{4} x_n^{-3}, \qquad n = 0, 1, \ldots.$$

Use this developed formula to find the second approximation to the 4^{th} root of 16, taking an initial approximation $x_0 = 1.5$. Compute the relative error. (6 Marks)

Question 3: The equation $x^3 e^{2x} = 0$ has a root $\alpha = 0$. Show that quadratic convergent method for finding approximation of this root is

$$x_{n+1} = \frac{2x_n^2}{(3 + 2x_n)}, \qquad n = 0, 1, \ldots.$$

Use this method to find x_2 using $x_0 = 0.1$. Find the rate of convergence of the developed iterative formula. (7 Marks)

Question 4: Use Simple Gauss-elimination method to find the many solutions of the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & -1 & \alpha \\ -1 & 2 & -\alpha \\ \alpha & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

by using the suitable value of the α. (6 Marks)

King Saud University: Mathematics Department Math-254
Second Semester 1447-48 H Solution of First Midterm Examination
Time: 90 Mins. Marks: 25

Question 1: Show that the following iterative sequences are the fixed point forms to the cubic root of a positive number N

$$\begin{aligned} x_{n+1} &= g_1(x_n) = Nx_n^{-2}; \qquad n = 0, 1, \ldots, \\ x_{n+1} &= g_2(x_n) = \sqrt{\frac{N}{x_n}}; \qquad n = 0, 1, \ldots. \end{aligned}$$

Use the convergent sequence to find the third approximation of the cubic root of 8 taking $x_0 = 1.5$ and then compute the absolute error. (6 Marks)

Solution. Given $x = N^{1/3}$ and it can be written as

$$x^3 - N = f(x) = 0,$$

which is equivalent to

$$x = \frac{N}{x^2} = Nx^{-2} = g_1(x),$$

and

$$x^2 = \frac{N}{x}, \quad \text{or} \quad x = \sqrt{\frac{N}{x}} = g_2(x).$$

So we have the given iterative schemes

$$\begin{aligned} x_{n+1} &= g_1(x_n) = Nx_n^{-2}; \qquad n = 0, 1, \ldots, \\ x_{n+1} &= g_2(x_n) = \sqrt{\frac{N}{x_n}}; \qquad n = 0, 1, \ldots. \end{aligned}$$

From the first iterative scheme, we have

$$g_1(x) = Nx^{-2} \qquad \text{and} \qquad g_1'(x) = -2Nx^{-3}.$$

Since $\alpha = x = N^{1/3}$, therefore

$$g_1'(\alpha) = -2N\alpha^{-3} \quad \text{and} \quad g_1'(N^{1/3}) = -2N(N^{1/3})^{-3} = -2NN^{-1} = -2.$$

Thus, $|g_1'(N^{1/3})| = |-2| = 2 > 1$, which shows that the first iterative scheme is diverges.
Similarly, from the second iterative scheme, we have

$$g_2(x) = \sqrt{\frac{N}{x}} = N^{1/2}x^{-1/2} \qquad \text{and} \quad g_2'(x) = -\frac{1}{2}N^{1/2}x^{-3/2}.$$

Since $\alpha = x = N^{1/3}$, therefore

$$g_2'(\alpha) = -\frac{1}{2}N^{1/2}(\alpha)^{-3/2} \quad \text{and} \quad g_2'(N^{1/3}) = -\frac{1}{2}N^{1/2}(N^{1/3})^{-3/2} = -\frac{1}{2}N^{1/2}N^{-1/2} = -\frac{1}{2}.$$

Thus, $|g_2'(N^{1/3})| = |-\frac{1}{2}| = \frac{1}{2} < 1$, which shows that the second iterative scheme is converges.
So to use the second iterative scheme to find the third approximation of $8^{1/3}$, using $N = 8$ and

$x_0 = 1.5$, we get

$$\begin{aligned} x_1 &= g_2(x_0) = \sqrt{\frac{N}{x_0}} = 2.3094; \\ x_2 &= g_2(x_1) = \sqrt{\frac{N}{x_1}} = 1.8612; \\ x_3 &= g_2(x_2) = \sqrt{\frac{N}{x_2}} = 2.0732. \end{aligned}$$

Thus

$$|8^{1/3} - x_3| = |2.0000 - 2.0732| = 0.0732,$$

is the absolute error. •

Question 2: Show that the Newton's iterative formula for evaluating 4^{th} root of a positive number N is

$$x_{n+1} = \frac{3}{4}x_n + \frac{N}{4}x_n^{-3}, \qquad n = 0, 1, \ldots.$$

Use this developed formula to find the second approximation to the 4^{th} root of 16, taking an initial approximation $x_0 = 1.5$. Compute the relative error. (6 Marks)

Solution. We shall compute $x = N^{1/4}$ by finding a positive root for the nonlinear equation

$$x^4 - N = 0,$$

where $N > 0$. Thus

$$f(x) = x^4 - N \quad \text{and} \quad f'(x) = 4x^3.$$

Hence, assuming an initial estimate to the root, say, $x = x_n$ and by using Newton's iterative formula, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^4 - N)}{4x_n^3} = x_n - \frac{x_n^4}{4x_n^3} + \frac{N}{4x_n^3} = (1 - \frac{1}{4})x_n + \frac{N}{4}x_n^{-3},$$

which gives Newton's formula for approximating 4^{th} root of a positive number N

$$x_{n+1} = \frac{3}{4}x_n + \frac{N}{4}x_n^{-3}, \quad n = 0, 1, \ldots.$$

Since we want the approximations of the 4^{th} root of number 16, so we take $N = 16$. Given the initial approximation $x_0 = 1.5$, then by using the developed iterative formula, we get

$$x_1 = \frac{3}{4}x_0 + \frac{N}{4}x_0^{-3} = 2.3102,$$

$$x_2 = \frac{3}{4}x_1 + \frac{N}{4}x_1^{-3} = 2.0359,$$

the required second approximation of the 4^th root of number 16.
Now to calculate the relative error as

$$|E| = \frac{|(16)^{1/4} - x_2|}{|(16)^{1/4}|} = \frac{|2.0000 - 2.0359|}{2} = 0.012.$$

•

Question 3: The equation $x^3e^{2x} = 0$ has a root $\alpha = 0$. Show that quadratic convergent method for finding approximation of this root is

$$x_{n+1} = \frac{2x_n^2}{(3 + 2x_n)}, \qquad n = 0, 1, \ldots.$$

Use this method to find x_2 using $x_0 = 0.1$. Find the rate of convergence of the developed iterative formula. (7 Marks)

Solution. Given $f(x) = x^3e^{2x}$ and $\alpha = 0$ is a root of $f(x) = 0$, so

$$f(0) = (0)^3e^0 = 0,$$

now we have to check its type (simple or multiple), we do have to check first derivative $f'(x)$ of $f(x)$ at $\alpha = 0$ as

$$f'(x) = (3x^2 + 2x^3)e^{2x}, \quad \text{and} \quad f'(0) = 0.$$

Thus the given root is multiple and we have to find its multiplicity as

$$\begin{array}{rclrcl} f''(x) & = & (6x + 12x^2 + 4x^3)e^{2x}, & f''(0) & = & 0, \\ f'''(x) & = & (6 + 36x + 36x^2 + 8x^3)e^{2x}, & f'''(0) & = & 6 \neq 0, \end{array}$$

so the function has zero of multiplicity 3. So quadratic convergent method is modified Newton's iterative formula, and by using it, we get

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} = x_n - 3\frac{x^3e^{2x}}{x^2(3 + 2x)e^{2x}} = x_n - 3\frac{x_n}{(3 + 2x_n)}, \quad n \geq 0.$$

Now evaluating this at the give approximation $x_0 = 0.1$, gives

$$x_1 = x_0 - 3\frac{x_0}{(3 + 2x_0)} = 0.0062 \quad \text{and} \quad x_2 = x_1 - 3\frac{x_1}{(3 + 2x_1)} = 2.5934 \times 10^{-5}.$$

The fixed point form of the developed modified Newton's formula is

$$x_{n+1} = g(x_n) = x_n - 3\frac{x_n}{(3 + 2x_n)},$$

where

$$g(x) = x - \frac{3x}{(3 + 2x)}.$$

By taking first derivative, we have

$$g'(x) = 1 - \left(\frac{(3 + 2x)(3) - 3x(2)}{(3 + 2x)^2}\right) = 1 - \left(\frac{9}{(3 + 2x)^2}\right), \quad g'(0) = 1 - 1 = 0,$$

and the second derivative of $g(x)$, gives

$$g''(x) = 0 - 9(-2)(2)(3 + 2x)^{-3} = \frac{36}{(3 + 2x)^3}, \quad g''(0) = 4/3 \neq 0.$$

Thus the rate of convergence of the developed iterative formula is two (2), means, the formula converges quadratically to the given root. •

Question 4: Use Simple Gauss-elimination method to find the many solutions of the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & -1 & \alpha \\ -1 & 2 & -\alpha \\ \alpha & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

by using the suitable value of the α. (6 Marks)

Solution. Using $m_{21} = -1$, $m_{31} = \alpha$ and $m_{32} = \dfrac{1+\alpha}{1}$, gives matrix form

$$[A|b] = \begin{pmatrix} 1 & -1 & \alpha & 1 \\ -1 & 2 & -\alpha & 1 \\ \alpha & 1 & 1 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & \alpha & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1+\alpha & 1-\alpha^2 & -1-\alpha \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & \alpha & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1-\alpha^2 & -3-3\alpha \end{pmatrix}.$$

Obviously, the original set of equations has been transformed to an upper-triangular form.
So if $1 - \alpha^2 \neq 0$, then we have the unique solution of the given system while for $\alpha = \pm 1$, we have no unique solution. If $\alpha = -1$, then we have infinitely many solution because third row of above augmented matrix get the form

$$0x_1 + 0x_2 + 0x_3 = 0,$$

while for $\alpha = 1$, we have

$$0x_1 + 0x_2 + 0x_3 = -6,$$

which is not possible, so no solution.
Thus taking suitable value of $\alpha = -1$, we have upper-triangular system of the form

$$\begin{array}{ccccccc} x_1 & - & x_2 & - & x_3 & = & 1 \\ & & x_2 & & & = & 2 \end{array}$$

Performing backward substitution and using $x_3 = t \in R,\ t \neq 0$, yields, $x_2 = 2$ and $x_1 = 3 + t$, the required many solutions of the given system using $\alpha = -1$. •