King Saud University: Mathematics Department MATh-254
Second Semester 1446-47 H Final Examination
Maximum Marks = 40 Time: 180 mins.

Name of the Student:________________ I.D. No. ________

Name of the Teacher:________________ Section No. ________

Note: Check the total number of pages are Six (6).
(15 Multiple choice questions and Two (2) Full questions)

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one $(2 \times 15 = 30)$

Ps. : Mark {A, B, C or D} for the correct answer in the following box.

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A,B,C,D															

Quest. No.	Marks Obtained	Marks for Questions
Q. 1 to Q. 15		30
Q. 16		5
Q. 17		5
Total		40

Question 1: The number of bisections required to solve the equation $x^3 - x^2 = 1$ in $[1, 2]$ accurate to within 10^{-6} is:

(A) 15 (B) 11 (C) 20 (d) None of these

Question 2: When using Newton's method with $x_0 = 1$ for solving the equation $\cos(\pi x - \pi) - x = 0$, the first approximation x_1 is:

(A) 1.55 (B) 1.00 (C) 1.35 (D) None of These

Question 3: Which of the following sequences will converge faster to $\sqrt{5}$:

(A) $x_{n+1} = x_n + 1 - \frac{x_n^2}{5}$ (B) $x_{n+1} = \frac{1}{3}[3x_n + 1 - \frac{x_n^2}{5}]$ (C) $x_{n+1} = \frac{5}{x_n}$ (D) None of These

Question 4: The rate of convergence the iterative scheme $x_{n+1} = \frac{1}{2}(x_n^2 + 1) - \ln x_n$, $n \geq 0$ to $\alpha = 1$ is:

(A) Order 2 (B) Order 3 (C) Order 1 (D) None of These

Questions (5 - 7) are concerned with Linear System $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0.5 \\ -2 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 0.5 & -0.25 \\ 1 & 0.5 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Question 5: In the LU factorization with $u_{ii} = 1$, $i = 1, 2$ of the matrix A, the matrix L is given by:

(A) $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$ (D) None of These

Question 6: For the given linear system, the l_∞-norm of the Jacobi iteration matrix T_J is equal to:

(A) 1.0 (B) 0.5 (C) 2.0 (D) None of these

Question 7: The relative error $\frac{\|\mathbf{x} - \mathbf{x}^*\|_\infty}{\|\mathbf{x}\|_\infty}$, where $x^* = [2, 0]^T$ be the approximation of the given linear system, is bounded by:

(A) 5.75 (B) 6.75 (C) 4.75 (D) None of These

Question 8: If $f(x) = x^2$ and the second order divided difference $f[\alpha, 2, 2] = 3$. Then the value of α is:

(A) 1 (B) 3 (C) 2 (D) None of These

Question 9: Using the Newton's interpolating polynomial of degree 2 for $f(x) = x \ln x + e^{-x}$ on the interval $[2, 4]$ with the points $x_0 = 2.0$, $x_1 = 3.0$, $x_2 = 4.0$, an error bound for the approximation of $f(2.4226)$ is:

(A) 0.0247 (B) 0.00247 (C) 0.0427 (D) None of These

Question 10: When using simple trapezoidal rule for approximating the integral $\int_1^{1.5} \frac{1}{x}\, dx$, we have the computed approximation:

(A) $\frac{7}{12}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{5}{14}$ **(D)** None of These

Question 11: If $f(0) = 3, f(1) = \frac{\alpha}{2}, f(2) = \alpha$, and Simpson's rule for $\int_0^2 f(x)\, dx$ gives 2, then the value of α is:

(A) 1.0 **(B)** 2.0 **(C)** -1.0 **(D)** None of These

Question 12: Using data points: $(0,1), (0.1,1.1), (0.2,1.3), (0.3,1.4), (0.4,1.5), (0.5,1.7)$, then the **best** approximate value of $f'(0)$ using 3-point difference formula is:

(A) 0.50 **(B)** 0.75 **(C)** 0.55 **(D)** None of These

Question 13: Let $f(x) = x \ln x + x$ and $x = 0.9, 1.3, 1.9, 2.1, 2.5, 3.2$. Then the absolute error for the approximation of $f''(1.9)$ is:

(A) 0.9300 **(B)** 0.0930 **(C)** 0.0093 **(D)** None of These

Question 14: Given initial-value problem $\frac{1}{x}y' - y^2 = 0,\ y(1) = 1$ and its exact solution is $y(x) = 2/(3 - x^2)$, then absolute error for the approximate value of $y(1.4)$ using Euler's method with $n = 2$ is:

(A) 0.3577 **(B)** 0.5377 **(C)** 0.3775 **(D)** None of These

Question 15: Given initial-value problem $e^y y' - e^x = 0,\ y(0) = 1$, the approximate value of $y(1)$ using Taylor's method of order 2 with $n = 2$ is:

(A) 1.4983 **(B)** 1.4893 **(C)** 1.4993 **(D)** None of These

Question 16: If $f(x) = x^2+\cos 2x$ (x is in radian) and x-values are $\{-0.5, 0.0, 0.3, 0.5, 0.6, 1.0\}$. Use the quadratic Lagrange interpolating polynomial for equally spaced data points to find the best approximation of $0.16 + \cos 0.8$. Compute an error bound and the absolute error.

Question 17: Find the approximation of $\int_0^{1.25} f(x)\,dx$ by using the following set of data points using the best integration rule:

x	0.0	0.1	0.25	0.31	0.4	0.5	0.6	0.75	0.8	0.91	1.0	1.1	1.25
$f(x)$	0	0.1098	0.3099	0.4012	0.5494	0.7294	0.9246	1.2441	1.3574	1.6176	1.8415	2.1012	2.5115

The function tabulated is $f(x) = x^2 + \sin x$ (x is in radian), compute the absolute error and the number of subintervals approximate the given integral to within accuracy of 10^{-3} ?

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A,B,C,D	A	C	B	C	A	B	C	A	B	A	C	C	B	A	A

The Answer Tables for Q.1 to Q.15 : (MATh) Marks: 2 for each one $(2 \times 15 = 30)$

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Question 16: If $f(x) = x^2 + \cos 2x$ (x is in radian) and x-values are $\{-0.5, 0.0, 0.3, 0.5, 0.6, 1.0\}$. Use the quadratic Lagrange interpolating polynomial for equally spaced data points to find the best approximation of $0.16 + \cos 0.8$. Compute an error bound and the absolute error.

Solution. Since the given function is $f(x) = x^2 + \cos 2x$, so by taking $2x = 0.8$, we have $x = 0.4$, therefore, the best points for the equally spaced data points Lagrange quadratic polynomial are, $x_0 = 0.0, x_1 = 0.3$ and $x_2 = 0.6$ with $h = 0.3$:

x	0.0	0.3	0.6
$f(x)$	1.0000	0.9153	0.7224

Consider the quadratic Lagrange interpolating polynomial as

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{1}$$

$$f(0.4) \approx p_2(0.4) = L_0(0.4)(1.0000) + L_1(0.4)(0.9153) + L_2(0.4)(0.7224). \tag{2}$$

The Lagrange coefficients can be calculate as follows:

$$L_0(0.4) = \frac{(0.4-0.3)(0.4-0.6)}{(0.0-0.3)(0.0-0.6)} = -0.1111,$$

$$L_1(0.4) = \frac{(0.4-0.0)(0.4-0.6)}{(0.3-0.0)(0.3-0.6)} = 0.8889,$$

$$L_2(0.4) = \frac{(0.4-0.0)(0.4-0.3)}{(0.6-0.0)(0.6-0.3)} = 0.2222.$$

Putting these values of the Lagrange coefficients in (2), we have

$$f(0.4) \approx p_2(0.4) = (-0.1111)(1.0000) + (0.8889)(0.9153) + (0.7224)(0.2222) = 0.8630,$$

which is the required approximation of the exact solution $f(0.4) = 0.16 + \cos 0.8 \approx 0.8567$. To compute an error bound for the approximation of the given function in the interval $[0.0, 0.6]$, we use the following quadratic error formula

$$|f(x) - p_2(x)| \leq \frac{Mh^3}{9\sqrt{3}}.$$

As

$$M = \max_{0.0 \leq x \leq 0.6} |f^{(3)}(x)|,$$

and the first three derivatives are

$$f'(x) = 2x - 2\sin 2x, \qquad f''(x) = 2 - 4\cos 2x, \qquad f^{(3)}(x) = 8\sin 2x,$$

$$M = \max_{0.0 \leq x \leq 0.6} \left| 8\sin 2x \right| = 7.4563.$$

Hence

$$|f(0.4) - p_2(0.4)| \leq \frac{(7.4563)(0.3^3)}{9\sqrt{(3)}} = 0.0129,$$

which is desired error bound. Also, we have

$$|f(0.4) - p_2(0.4)| = |(0.16 + \cos 0.8) - 0.8630| = |0.8567 - 0.8630| = 0.0063,$$

the desired absolute error. •

Question 17: Find the approximation of $\int_0^{1.25} f(x)\,dx$ by using the following set of data points using the best integration rule:

x	0.0	0.1	0.25	0.31	0.4	0.5	0.6	0.75	0.8	0.91	1.0	1.1	1.25
$f(x)$	0	0.1098	0.3099	0.4012	0.5494	0.7294	0.9246	1.2441	1.3574	1.6176	1.8415	2.1012	2.5115

The function tabulated is $f(x) = x^2 + \sin x$ (x is in radian), compute the absolute error and the number of subintervals approximate the given integral to within accuracy of 10^{-3} ?

Solution. Choosing the equally spaced data points, $0.0, 0.25, 0.5, 0.75, 1.0, 1.25$, gives $h = 0.25$, we select the following table:

x	0.0	0.25	0.5	0.75	1.0	1.25
$f(x)$	0.0000	0.3099	0.7294	1.2441	1.8415	2.5115

so the composite Trapezoidal rule for six points (or $n = 5$ (or $h = 0.25$) can be written as

$$\int^1 .25_0\ f(x)\,dx \approx T_5(f) = \frac{h}{2}\Big[f(x_0) + 2\Big(f(x_1) + f(x_2) + f(x_3) + f(x_4)\Big) + f(x_5)\Big],$$

$$\int^1 .25_0\ f(x)\,dx \approx 0.125\Big[0.0 + 2(0.3099 + 0.7294 + 1.2441 + 1.8415) + 2.5115\Big] = 1.3452.$$

We can easily computed the exact value of the given integral as

$$I(f) = \int_0^{1.25} x^2 + \sin x\,dx = (x^3/3 - \cos x)\Big|_0^{1.25} = 1.3357.$$

Thus the absolute error $|E|$ in our approximation is given as

$$|E| = |I(f) - T_5(f)| = |1.3357 - 1.3452| = 0.0095.$$

The first two derivatives of the function $f(x) = x^2 + \sin x$ can be obtain as

$$f'(x) = 2x + \cos x \quad \text{and} \quad f''(x) = 2 - \sin x.$$

Since $\eta(x)$ is unknown point in $(0, 1.25)$, therefore, the bound $|f''|$ on $[0, 1.25]$ is

$$M = \max_{0 \le x \le 1.25} |f''(x)| = \max_{0 \le x \le 1.25} |2 - \sin x| = 2.0,$$

at $x = 0$. The error formula of the composite Trapezoidal rule is

$$|E_{T_5}(f)| \le \frac{h^2(b-a)}{12} M,$$

or

$$|E_{T_5}(f)| \le \frac{(b-a)^2(b-a)}{12n^2} M, \qquad h = (b-a)/n.$$

To find the minimum subintervals for the given accuracy, we use the error formula such that

$$|E_{T_n}(f)| \le \frac{(1.25-0)^3}{12n^2}(2.0) \le 10^{-3},$$

then solving for n,

$$n^2 \ge \frac{(1.25)^3(2.0)10^3}{12}, \quad \text{or} \quad n \ge \sqrt{\frac{(1.25)^3(2.0)10^3}{12}},$$

we obtain, $n \ge 18.0422$. Hence to get the required accuracy, we need 19 subintervals. •