[Solution Key] Sem 472 / MTE2 / MATH-244

Question 1: V.1: Correct choices: a, c, b, b, b, b, b, b, a, b.

V.2: Correct Choices: d, b, b, a, a, a, a, a, d, a.

V.3: Correct Choices: c, a, a, d, d, d, d, d, c, d.

V.4: Correct Choices: b, d, d, c, c, c, c, c, b, c.

Question: a) $\{(1,1,1), (1,0,0), (0,1,0), (0,0,1)\}$

b) $\alpha(2u_1) + \beta(u_1 - 2u_2) + \gamma(2u_1 + u_3) = 0$

$\Rightarrow (2\alpha + \beta + 2\gamma)u_1 - 2\beta u_2 + \gamma u_3 = 0$

$\Rightarrow 2\alpha + \beta + 2\gamma = 0, 2\beta = 0, \gamma = 0$ (since $\{u_1, u_2, u_3\}$ is a basis of V)

$\Rightarrow \alpha = \beta = \gamma = 0$. Hence, $\{2u_1, u_1 - 2u_2, 2u_1 + u_3\}$ is linearly independent. Moreover, $\{2u_1, u_1 - 2u_2, 2u_1 + u_3\}$ consists of three vectors and dimension of V is 3. Thus, $\{2u_1, u_1 - 2u_2, 2u_1 + u_3\}$ is a basis of V.

c) Clearly, $V_3 = V_2 - V_1$ and $\{V_1, V_2\}$ is linearly independent (since $\alpha_1 V_1 + \alpha_2 V_2 = (0,0,0) \Rightarrow \alpha_1 + 2\alpha_2 = 0$ and $\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2 = 0$). Hence, $B := \{V_1, V_2\}$ is a basis of span(A).

Next, since $\det\left(\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right) = -1 \neq 0$, the set $\{V_1, V_2, (0,1,0)\}$ is a basis C of $\mathbb{R}^3$ such that $B \subseteq C$.

Question 3 a) (i) $[u_1\ u_2\ u_3 \mid V] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow [V]_B = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

(ii) $[V_1\ V_2\ V_3 \mid u_1\ u_2\ u_3] = \left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & 1 & \frac{3}{2} & 1 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & 0 \end{array}\right]$.

So that $P_{B \to C} = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ 1 & \frac{3}{2} & 1 \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$. Hence, $[V]_C = P_{B \to C}[V]_B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

b) Since $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (REF),

we get rank and nullity of the given matrix are 4 and 0, respectively.

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[V. 2]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 472 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hrs.**

Name: **ID:** **Section:** **Signature:**

Note: Please fill in the above columns. Calculators are not allowed.

Question 1: [Marks: 10]

Which of the given choices are correct? write the correct choice number in the following table:

(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)

(i) If $B = \{u,v,w\}$ is a basis of $\mathbb{R}^3$ and $ru + sv + tw = 0$ for some scalars r, s, t, then:
a) $u = v = w$ b) B is linearly dependent c) $r = u, s = v, t = w$ d) $r = s = t = 0$

(ii) If a subset S of the vector space $\mathbb{R}^3$ spans $\mathbb{R}^3$, what must be true?
a) S is linearly independent.
b) S contains at least three linearly independent vectors.
c) All vectors in S are non-zero.
d) S contains exactly three vectors.

(iii) If A is a matrix of size 5×7, then its:
a) rows are linearly independent b) columns are linearly dependent c) rank is 5 d) rank is 7

(iv) If $S = \{u,v,w\}$ is an ordered basis of a vector space V, then the coordinate vector $[u]_S$ with respect to the ordered basis S is:
a) $\begin{bmatrix}1\\0\\0\end{bmatrix}$ b) $\begin{bmatrix}1\\1\\1\end{bmatrix}$ c) $\begin{bmatrix}0\\0\\1\end{bmatrix}$ d) $\begin{bmatrix}0\\0\\0\end{bmatrix}$

(v) If $S = \{u,v,w,x\}$ is a subset of the vector space $\mathbb{R}^3$, then S must be:
a) linearly dependent b) linearly independent c) a basis of $\mathbb{R}^3$ d) a subspace of $\mathbb{R}^3$

(vi) Which of the following sets is a subspace of the vector space $\mathbb{R}^3$?
a) $\{(x,y,z): x + y + z = 0\}$ b) $\{(x,y,z): x^2 + y^2 + z^2 = 1\}$ c) $\{(x,y,z): x > 0\}$ d) $\{(x,y,z): x + y + z = 1\}$

(vii) If a subset $S \neq \{0\}$ of a vector space V is linearly dependent, then:
a) At least one vector in S is a linear combination of other vectors in S.
b) One vector in S must be 0.
c) The set S must span the vector space V.
d) No vector in S can be written as a linear combination of other vectors in S.

(viii) Any transition matrix P must satisfy that:
a) P is invertible b) P is symmetric c) P is diagonal d) $\det(P) = 0$

(ix) If A is a 3×6 matrix with linearly independent rows, then:
a) $\text{rank}(A) = 4$ b) $\text{rank}(A) = 5$ c) $\text{rank}(A) = 6$ d) $\text{rank}(A) = 3$

(x) If a matrix A is of size 4×6 such that $\dim(\text{col}(A)) = 3$, then $\dim(N(A))$ is equal to:
a) 3 b) 0 c) ≤ 2 d) 1

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Question 2: [Marks: 2 + 2 + 3]

a) Give an example of a linearly dependent subset of the vector space $\mathbb{R}^3$ which spans $\mathbb{R}^3$.

b) Let $\{u_1, u_2, u_3\}$ be a basis of a vector space V. Determine whether the set $\{2u_1, u_1 - 2u_2, 2u_1 + u_3\}$ is a basis of V or not.

c) Let $A = \{v_1 = (1,1,1), v_2 = (2,1,1), v_3 = (1,0,0)\}$. Find a basis B of span(A), and then find a basis C of $\mathbb{R}^3$ such that $B \subseteq C$.

Question 3: [Marks: (2+3) + 3]

a) Consider the two ordered bases $B = \{u_1 = (0,1,1), u_2 = (1,1,1), u_3 = (1,0,1)\}$ and $C = \{v_1 = (0,1,-1), v_2 = (1,1,0), v_3 = (-1,0,1)\}$ of the vector space $\mathbb{R}^3$, and the vector $v = (1,2,1) \in \mathbb{R}^3$.

(i) Find the coordinate vector $[v]_B$ of the vector v with respect to the ordered basis B.

(ii) Construct the transition matrix $P_{B \to C}$ from the basis B to the basis C, and then use it to find the coordinate vector $[v]_C$ of the vector v with respect to the ordered basis C.

b) Find the rank and nullity of the matrix $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}$.

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