Math 244 Linear Algebra
Semester 481, Exam 2
Time allowed: 90 minutes.

25 pts

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

NAME: **Student ID:** **Section Number:**

Note: Calculators are not allowed.

Question	1	2	3	4	5	6	7	8	9	10	
Answer											10 pts

Question 1 Write the letter of the correct choice in the table provided.

(1) Let A be a 5×4 matrix such that $\text{nullity}(A) = 1$. Then:

a) $\text{rank}(A) = 4$ b) $\text{rank}(A^T) = 4$ c) $\text{nullity}(A^T) = 1$ d) $\text{nullity}(A^T) = 2$

(2) Let $W = \text{span}\{(1,1,0), (0,0,1)\}$. Which of the following vectors belongs to W?

a) $(1,1,2)$ b) $(1,-1,0)$ c) $(0,1,0)$ d) $(-1,0,0)$

(3) Let $E = \text{span}\{1+x, 1+x^2\}$. Which of the following polynomials belongs to E?

a) 1 b) x c) $2+x+x^2$ d) x^2

(4) Find the set of all values of k such that $S = \{(1,k), (k,4)\}$ is a basis for $\mathbb{R}^2$.

a) $\{-2,2\}$ b) $\mathbb{R} - \{-2,2\}$
c) $\{1,4\}$ d) $\mathbb{R}$

(5) Let $B = \{v_1, v_2\}$ and $C = \{w_1, w_2\}$, where $w_1 = (1,1)$ and $w_2 = (1,-1)$, be two ordered bases for $\mathbb{R}^2$. If ${}_C P_B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ is the transition matrix from B to C, then:

a) $B = \{(3,-1), (0,-2)\}$ b) $B = \{(1,-1), (2,1)\}$
c) $B = \{(0,2), (3,1)\}$ d) $B = \{(3,0), (-1,-2)\}$

(6) Let $B = \{u_1, u_2, u_3, u_4\}$ be a basis of a vector space V. Which of the following sets is also a basis of V?

a) $\{u_1 + u_2, u_1, u_2, u_3, u_4\}$ b) $\{u_1, u_2, u_3, u_1 + u_2\}$
c) $\{u_1 + u_2, u_3, u_4\}$ d) $\{u_1 + u_2, u_2, u_3, u_4\}$

(7) Let A be a 5×5 matrix with $\text{nullity}(A) = 2$. Which of the following statements is true?

a) The rows of A are linearly dependent b) $\det(A) \neq 0$
c) The columns of A span $\mathbb{R}^5$ d) $Ax = b$ is consistent for every $b \in \mathbb{R}^5$

(8) Which of the following subsets of $\mathbb{R}^3$ is a vector subspace?

a) $\{(x,y,z) \in \mathbb{R}^3 : xy = 0\}$ b) $\{(s+t, s-t, 2s) : s,t \in \mathbb{R}\}$
c) $\{(x,y,z) \in \mathbb{R}^3 : x + y + z = 1\}$ d) $\{(x,y,z) \in \mathbb{R}^3 : x \geq 0\}$

(9) Let u, v be linearly independent vectors in a vector space V and let $w = 2u - v$. Then $\dim(\text{span}\{u, v, w\})$ is:

a) 2 b) 1 c) 3 d) 0

(10) Let A be a 3×4 matrix with $\text{rank}(A) = 3$. Then:

a) $Ax = b$ has a unique solution for every $b \in \mathbb{R}^3$
b) The columns of A are linearly independent
c) $Ax = b$ has infinitely many solutions for every $b \in \mathbb{R}^3$
d) $\text{nullity}(A) = 2$

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Question 2 Let $B = \{v_1, v_2, v_3\}$ and $C = \{w_1, w_2, w_3\}$ be two ordered bases of a vector space V such that

2.5 pts

$$v_1 - 4w_3 = w_1 + w_2$$
$$v_2 + 2w_1 = 3w_2$$
$$v_3 + w_2 = w_1 + w_3$$

Find ${}_C P_B$, the transition matrix from B to C.

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Question 3 Find a basis for the subspace $E = \{a + bx - (2a + 3b)x^2 : a, b \in \mathbb{R}\}$ of P_2.

2.5 pts

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Question 4 Let $S = \{v_1, v_2, v_3\}$ be a basis of a vector space V, and let

$$u = v_1 + v_2 + v_3 \quad \text{and} \quad w = -v_1 + v_2 + v_3.$$

Show that u and w are linearly independent.

2.5 pts

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Question 5 Let $A = \begin{bmatrix} 1 & -3 & 1 & 1 \\ -1 & 3 & 2 & -4 \\ 2 & -6 & 1 & 3 \\ 1 & -3 & -1 & 3 \\ 1 & -3 & 3 & -1 \end{bmatrix}$ and $\text{rref}(A) = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

7.5 pts

(i) Find a basis for the row space $\text{row}(A)$ and a basis for the null space $\text{null}(A)$.

(ii) Find a basis B for the column space $\text{col}(A)$, then find the coordinate vector $[(1,-4,3,3,-1)]_B$.

(iii) Find $\text{rank}(A^T)$ and $\text{nullity}(A^T)$.

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Math 244 Linear Algebra, Semester 481, Exam 2
Solution Key

Question 1 (10 pts)

Identify each paper's form from Q1(1): the correct choice $\text{nullity}(A^T) = 2$ is a) in Form A, b) in Form B, c) in Form C and d) in Form D.

Question	1	2	3	4	5	6	7	8	9	10
Form A	a	d	d	c	b	c	b	a	d	b
Form B	b	c	b	d	a	a	d	c	b	a
Form C	c	b	a	a	d	b	c	d	c	d
Form D	d	a	c	b	c	d	a	b	a	c

Correct choices and justifications (the same in every form; only the letter changes)

(1) **Correct choice:** $\text{nullity}(A^T) = 2$. Since $\text{rank}(A) = 4 - \text{nullity}(A) = 3$, we get $\text{rank}(A^T) = 3$, and since A^T is 4×5, $\text{nullity}(A^T) = 5 - 3 = 2$.

(2) **Correct choice:** $(1, 1, 2)$. Here $W = \{(a, a, b) : a, b \in \mathbb{R}\}$ and $(1, 1, 2) = (1, 1, 0) + 2(0, 0, 1)$. None of the other vectors has equal first and second entries.

(3) **Correct choice:** $2 + x + x^2$, since $2 + x + x^2 = (1 + x) + (1 + x^2)$. Every element $a(1 + x) + b(1 + x^2) = (a + b) + ax + bx^2$ has constant term equal to the sum of its other two coefficients, which fails for 1, x and x^2.

(4) **Correct choice:** $\mathbb{R} - \{-2, 2\}$. S is a basis for $\mathbb{R}^2$ if and only if $\begin{vmatrix} 1 & k \\ k & 4 \end{vmatrix} = 4 - k^2 \neq 0$, that is, $k \neq \pm 2$.

(5) **Correct choice:** $B = \{(0, 2), (3, 1)\}$. The columns of ${}_C P_B$ are $[v_1]_C = (1, -1)$ and $[v_2]_C = (2, 1)$, so $v_1 = w_1 - w_2 = (0, 2)$ and $v_2 = 2w_1 + w_2 = (3, 1)$.

(6) **Correct choice:** $\{u_1 + u_2, u_2, u_3, u_4\}$. These four vectors span V because $u_1 = (u_1 + u_2) - u_2$, and four vectors spanning the 4-dimensional space V form a basis. Each of the other sets has five vectors, contains u_1, u_2 and $u_1 + u_2$ together, or has only three vectors.

(7) **Correct choice:** The rows of A are linearly dependent. Here $\text{rank}(A) = 5 - 2 = 3 < 5$, so A is not invertible: $\det(A) = 0$, the columns do not span $\mathbb{R}^5$, and $Ax = b$ is not consistent for every b.

(8) **Correct choice:** $\{(s + t, s - t, 2s) : s, t \in \mathbb{R}\}$. It equals $\text{span}\{(1, 1, 2), (1, -1, 0)\}$. The others fail: $xy = 0$ is not closed under addition, since $(1, 0, 0) + (0, 1, 0) = (1, 1, 0)$; $x + y + z = 1$ does not contain $(0, 0, 0)$; and $x \geq 0$ is not closed under multiplication by -1.

(9) **Correct choice:** 2. Since $w = 2u - v \in \text{span}\{u, v\}$, we have $\text{span}\{u, v, w\} = \text{span}\{u, v\}$, which has dimension 2 because u and v are linearly independent.

(10) **Correct choice:** $Ax = b$ has infinitely many solutions for every $b \in \mathbb{R}^3$. Since $\text{rank}(A) = 3$, $\text{col}(A) = \mathbb{R}^3$, so the system is always consistent; and $\text{nullity}(A) = 4 - 3 = 1 > 0$, so each system has infinitely many solutions. In particular, the four columns are linearly dependent and $\text{nullity}(A) \neq 2$.

Question 2 (2.5 pts)

Rearranging the given relations:

$$v_1 = w_1 + w_2 + 4w_3, \quad v_2 = -2w_1 + 3w_2, \quad v_3 = w_1 - w_2 + w_3.$$

Hence the coordinate vectors relative to C are

$$[v_1]_C = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \quad [v_2]_C = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}, \quad [v_3]_C = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

and these are the columns of the transition matrix from B to C:

$${}_C P_B = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 3 & -1 \\ 4 & 0 & 1 \end{pmatrix}.$$

Question 3 (2.5 pts)

Every element of E can be written as

$$a + bx - (2a + 3b)x^2 = a(1 - 2x^2) + b(x - 3x^2),$$

so $E = \text{span}\{1 - 2x^2, x - 3x^2\}$. The two polynomials are linearly independent, since neither is a scalar multiple of the other. Therefore a basis for E is $\{1 - 2x^2, x - 3x^2\}$.

Question 4 (2.5 pts)

Suppose $\alpha u + \beta w = 0$ for scalars α, β. Then

$$\alpha u + \beta w = (\alpha - \beta)v_1 + (\alpha + \beta)v_2 + (\alpha + \beta)v_3 = 0.$$

Since S is a basis, v_1, v_2, v_3 are linearly independent, so $\alpha - \beta = 0$ and $\alpha + \beta = 0$. Hence $\alpha = \beta = 0$, and u and w are linearly independent.

Question 5 (7.5 pts)

Let c_1, c_2, c_3, c_4 denote the columns of A. The leading 1s of $\text{rref}(A)$ are in columns 1 and 3, so $\text{rank}(A) = 2$.

(i) Row space: the nonzero rows of $\text{rref}(A)$ form a basis for $\text{row}(A)$:

$$\{(1, -3, 0, 2), (0, 0, 1, -1)\}.$$

Null space: $\text{rref}(A)x = 0$ gives $x_1 = 3x_2 - 2x_4$ and $x_3 = x_4$. With free variables $x_2 = s$ and $x_4 = t$,

$$x = s(3, 1, 0, 0) + t(-2, 0, 1, 1),$$

so a basis for $\text{null}(A)$ is $\{(3, 1, 0, 0), (-2, 0, 1, 1)\}$.

(ii) The pivot columns of A form a basis for $\text{col}(A)$:

$$B = \{c_1, c_3\} = \{(1, -1, 2, 1, 1), (1, 2, 1, -1, 3)\}.$$

The given vector is the fourth column of A: $(1, -4, 3, 3, -1) = c_4$. Column 4 of $\text{rref}(A)$ equals $2 \cdot$(column 1) $-$ (column 3), and row operations preserve linear relations among columns, so $c_4 = 2c_1 - c_3$. Therefore

$$[(1, -4, 3, 3, -1)]_B = (2, -1).$$

(iii) $\text{rank}(A^T) = \text{rank}(A) = 2$. Since A^T is 4×5, $\text{nullity}(A^T) = 5 - \text{rank}(A^T) = 3$.