[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS
Semester 462 / MATH-244 (Linear Algebra) / Mid-term Exam 1

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hrs.**

Question 1: [Marks: 5]

Determine whether the following statements are true or false and justify your answer:

(i) If A and B are symmetric matrices compatible for the product AB, then AB is also symmetric.

False: for example, $A=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, B=\begin{bmatrix}2 & 3\\ 3 & 2\end{bmatrix}$ are symmetric but $AB=\begin{bmatrix}5 & 5\\ -1 & 1\end{bmatrix}$ is not symmetric.

(ii) If the matrix A^2 is invertible, then A itself is invertible.

True: A is not invertible $\Rightarrow |A|=0 \Rightarrow |A^2|=|A|^2=0 \Rightarrow A^2$ is not invertible.

(iii) If the matrix $\begin{bmatrix}-4 & -3 & -3\\ 1 & 0 & 1\\ 4 & 4 & x\end{bmatrix}$ is its own adjoint, then $x=3$.

True: $adj\left(\begin{bmatrix}-4 & -3 & -3\\ 1 & 0 & 1\\ 4 & 4 & x\end{bmatrix}\right)=\begin{bmatrix}-4 & -3 & -3\\ 1 & 0 & 1\\ 4 & 4 & x\end{bmatrix} \Rightarrow x=c_{33}=(-1)^{3+3}M_{33}=\begin{vmatrix}-4 & -3\\ 1 & 0\end{vmatrix}=3.$

(iv) If square matrices A and B are compatible for the product AB, then $|AB|=|BA|$.

True: $|AB|=|A||B|=|B||A|=|BA|$.

(v) If $A=\begin{bmatrix}1 & 2\\ -5 & 1\end{bmatrix}$, then $(-\frac{1}{11},\frac{2}{11})$ is a solution of the equation $A^{-1}=xA+yI_2$.

True: $A^{-1}=\frac{1}{|A|}adj((A)=\frac{1}{11}\begin{bmatrix}1 & -2\\ 5 & 1\end{bmatrix}=-\frac{1}{11}\begin{bmatrix}1 & 2\\ -5 & 1\end{bmatrix}+\frac{2}{11}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=xA+yI_2.$

Question 2: [Marks: 4 + 3 + 3]

(a) Find the matrix A if $A^{-1}=\frac{1}{2}\begin{bmatrix}1 & -1 & 1\\ -1 & 3 & 1\\ 1 & 1 & -2\end{bmatrix}$.

Solution: $A=(A^{-1})^{-1}=2\begin{bmatrix}1 & -1 & 1\\ -1 & 3 & 1\\ 1 & 1 & -2\end{bmatrix}^{-1}=\frac{1}{5}\begin{bmatrix}7 & 1 & 4\\ 1 & 3 & 2\\ 4 & 2 & -2\end{bmatrix}.$

(b) Show that $\begin{vmatrix}1 & 1 & 1\\ b+c & c+a & a+b\\ -(b+c-a) & -(c+a-b) & -(a+b-c)\end{vmatrix}=0.$

Solution: $\begin{vmatrix}1 & 1 & 1\\ b+c & c+a & a+b\\ -(b+c-a) & -(c+a-b) & -(a+b-c)\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ b+c & c+a & a+b\\ a & b & c\end{vmatrix}$

$=\begin{vmatrix}1 & 1 & 1\\ a+b+c & b+c+a & c+a+b\\ a & b & c\end{vmatrix}=(a+b+c)\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ a & b & c\end{vmatrix}=0.$

(c) Let A be a square matrix of size n with $|A|=3$ and $|adj(A)|=27$. Find n.

Solution: $adj(A)=|A|A^{-1} \Rightarrow |adj(A)|=|A|^{n-1}$. Hence, $3^3=27=3^{n-1}$ gives $n=3+1=4$.

Question 3: [Marks: 3 + 3 + 4]

(a) Solve the matrix equation: $XA=B$, where $A=\begin{bmatrix}1 & 2 & -1\\ 2 & -1 & -1\\ -4 & 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 1 & 1\\ 8 & 1 & -5\\ 4 & 3 & -3\end{bmatrix}$.

Solution: Since $|A|=-3$, A^{-1} exists. $XA=B \Rightarrow X=BA^{-1}=\begin{bmatrix}-2 & 1 & 1\\ 8 & 1 & -5\\ 4 & 3 & -3\end{bmatrix}\left(\frac{1}{3}\begin{bmatrix}3 & 6 & 3\\ 2 & 1 & 1\\ 4 & 8 & 5\end{bmatrix}\right)=\begin{bmatrix}0 & -1 & 0\\ 2 & 3 & 0\\ 2 & 1 & 0\end{bmatrix}.$

(b) Solve the following system of linear equations by using Cramer's Rule:

$$\begin{aligned} x+3y+3z &= 1\\ x+3y+5z &= -1\\ x+2y-z &= 2. \end{aligned}$$

Solution: $|A|=\begin{vmatrix}1 & 3 & 3\\ 1 & 3 & 5\\ 1 & 2 & -1\end{vmatrix}=2$, $|A_x|=\begin{vmatrix}1 & 3 & 3\\ -1 & 3 & 5\\ 2 & 2 & -1\end{vmatrix}=-10$, $|A_y|=\begin{vmatrix}1 & 1 & 3\\ 1 & -1 & 5\\ 1 & 2 & -1\end{vmatrix}=6$ and $|A_z|=\begin{vmatrix}1 & 3 & 1\\ 1 & 3 & -1\\ 1 & 2 & 2\end{vmatrix}=-2.$

Hence. $x=\frac{|A_x|}{|A|}=-\frac{10}{2}=-5$, $y=\frac{|A_y|}{|A|}=\frac{6}{2}=3$ and $z=\frac{|A_z|}{|A|}=-\frac{2}{2}=-1$.

(c) Find the value of m for which the following system of linear equations admits a unique solution and then find this uniquely existing solution.

$$\begin{aligned} x+y+z &= 1\\ x+y+2z &= 0\\ 2x-y-z &= -1\\ x-2y+z &= m. \end{aligned}$$

Solution: $[A\!:\!B]=\begin{bmatrix}1 & 1 & 1 & : & 1\\ 1 & 1 & 2 & : & 0\\ 2 & -1 & -1 & : & -1\\ 1 & -2 & 1 & : & m\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 1 & : & 1\\ 0 & 1 & 1 & : & 1\\ 0 & 0 & 1 & : & -1\\ 0 & 0 & 0 & : & m+5\end{bmatrix}$. So, $m=-5$ gives a unique solution $x=0, y=2, z=-1$.

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