[RDraft]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

**Semester 472 / MATH-244 (Linear Algebra) / Mid-term Exam 1**

**Max. Marks: 25** **Max. Time: $1\frac{1}{2}$ hrs.**

**Note:** Calculators are not allowed.

**Question 1:** [Marks: 1+1+1+1+1]

Which of the given choices are correct?

(i) If $A$ is a square matrix and $A^2 = 0$ and $I$ is the identity matrix, then $(2A+I)^{-1}$ is equal to:

a) $I - A$ b) $I + A$ c) $I - 2A$ d) $2I - A$.

(ii) If $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then the unique solution of the equation $A^{-1} = \alpha A + \beta I$ is:

a) $\alpha = \frac{1}{10}, \beta = -\frac{1}{5}$ b) $\alpha = \frac{1}{10}, \beta = -\frac{3}{10}$ c) $\alpha = -\frac{1}{10}, \beta = \frac{1}{2}$ d) $\alpha = \frac{1}{10}, \beta = -\frac{1}{2}$.

(iii) Which of the following matrices is not an elementary matrix:

a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

(iv) If $A\,adj(A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then determinant of the matrix $A$ satisfies:

a) $|A| > 3$ b) $|A| < 5$ c) $|A| = 9$ d) $|A| = 27$.

(v) If $I$ is a row echelon form of the matrix $A$, then the linear system $AX = B$ would have:

a) infinitely many solutions b) no solutions c) unique solution d) more than one solutions.

**Question 2:** [Marks: 4+4+2]

a) Let A and B be two square matrices of the same size. Then:

i) Show that $(A+B)^2 = A^2 + 2AB + B^2$ if $(A+B)(A-B) = A^2 - B^2$.

ii) Give an example of matrices $A$ and $B$ for which $(A+B)^2 \neq A^2 + 2AB + B^2$.

b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Find all matrices $B$ such that $AB = I$,

c) Show that $\begin{vmatrix} a & a & b & b \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix} = (a+b)(a-b)^3$.

**Question 3:** [Marks: 6+4]

a) Find the value/s of $\alpha$ such that the following linear system:

$$\begin{aligned} x + 2y - z &= 2 \\ x + 2y - (\alpha^2 - 3)z &= \alpha \\ x - 2y + 3z &= 1 \end{aligned}$$

has:

(i) no solution (ii) unique solution (iii) infinitely many solutions.

b) Consider the following system of linear equations:

$$\begin{aligned} x + 2y - 3z &= 1 \\ x + y - z &= 2 \\ x + y + 2z &= 3. \end{aligned}$$

i) Find inverse $A^{-1}$ of the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ by using $adj(A)$.

ii) Use $A^{-1}$ to solve the above linear system.

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MATH244 / B472 / Solution Key of Mid-term Exam 1

Question 1: Correct choices: (1 mark for each part)

(i) c, (ii) c, (iii) b, (iv) b, (v) c

Question 2: a) i) $A^2+2AB+B^2 = (A+B)^2 = A^2+AB+BA+B^2$ gives $AB=BA$. (1)

Hence, $(A+B)(A-B) = A^2-AB+BA-B^2 = A^2-B^2$. (1)

ii) Taking $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\1&0\end{bmatrix}$, we have $(A+B)^2=\begin{bmatrix}5&4\\4&5\end{bmatrix}$, (2)

but $A^2+2AB+B^2=\begin{bmatrix}6&4\\4&4\end{bmatrix}$.

b) Since the matrix A is of size 2x3 and $AB=I$, we may assume that $B=\begin{bmatrix}x_1&y_1\\x_2&y_2\\x_3&y_3\end{bmatrix}$. Then, $\begin{bmatrix}1&1&1\\2&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1&1&1\\2&1&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$. (2)

These linear systems have solutions: $x_1=-1+s$, $x_2=2-2s$, $x_3=s$, $\forall s\in\mathbb{R}$ and $y_1=1+t$, $y_2=-1-2t$, $y_3=t$ $\forall t\in\mathbb{R}$, respectively. Thus, the required matrices are $B=\begin{bmatrix}-1+s&1+t\\2-2s&-1-2t\\s&t\end{bmatrix}$, $\forall s,t\in\mathbb{R}$. (2)

c) $\begin{vmatrix}a&a&b&b\\b&a&a&b\\b&b&a&a\\b&b&b&a\end{vmatrix}=\begin{vmatrix}a+b&a+b&a+b&a+b\\b&a&a&b\\b&b&a&a\\b&b&b&a\end{vmatrix}=(a+b)\begin{vmatrix}1&1&1&1\\b&a&a&b\\b&b&a&a\\b&b&b&a\end{vmatrix}=(a+b)\begin{vmatrix}1&1&1&1\\0&a-b&a&b\\0&0&a-b&a-b\\0&0&0&a-b\end{vmatrix}=(a+b)(a-b)^3$. (2)

Question 3: a) $[A|B]=\left[\begin{array}{ccc|c}1&2&-1&2\\1&2&3-\alpha^2&\alpha\\1&-2&3&1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&2&-1&2\\0&-4&4&-1\\0&0&2^2-\alpha^2&\alpha-2\end{array}\right]$. Hence, (3)

(i) $\alpha=-2$ (ii) $\alpha\in\mathbb{R}\setminus\{\pm2\}$ (iii) $\alpha=2$ (1+1+1)

b) ~~$[A|I]=\left[\begin{array}{ccc|ccc}1&2&-3&1&0&0\\1&1&-1&0&1&0\\1&1&2&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&-1&7/3&-1/3\\0&1&0&1&-5/3&2/3\\0&0&1&0&-1/3&1/3\end{array}\right]$. So, $A^{-1}=\frac{1}{3}\begin{bmatrix}-3&7&-1\\3&-5&2\\0&-1&1\end{bmatrix}$.~~

Hence, $\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{3}\begin{bmatrix}-3&7&-1\\3&-5&2\\0&-1&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}8/3\\-1/3\\1/3\end{bmatrix}$. (1.5)

Inverse of A by using adj(A):

$|A|=-3$ and The cofactor matrix $C=\begin{bmatrix}3&-3&0\\-7&5&1\\1&-2&-1\end{bmatrix}$. (1.5)

So that $A^{-1}=\frac{1}{|A|}\operatorname{adj}(A)=\frac{1}{|A|}C^T=\frac{-1}{3}\begin{bmatrix}3&-3&0\\-7&5&1\\1&-2&-1\end{bmatrix}^T$ (1)

$=\begin{bmatrix}-1&7/3&-1/3\\1&-5/3&2/3\\0&-1/3&1/3\end{bmatrix}$.

Correction in the solution key for the midterm exam 1 of MATH-244: ,there is an example given in the solution of Q.2/a)/ii): Actually, the 2nd row 2nd column entry of . (the matrix B is 1 (not 0

The proof given in the solution key is for the converse direction of what is required in the Q.1/a)/i): a proof in :the direction asked is as follows

A^2 - B^2= (A+B)(A-B) =A^2 +AB-BA- B^2 gives

,AB=BA. Hence

(A+B)^2=A^2+2AB+ B^2