[V. 1]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 472 / MATH-244 (Linear Algebra) / Final Exam

Max. Marks: 40 Time: 3 hours

Name: ID: Section: Signature:

Note: Please fill in the above columns. Calculators are not allowed.

Question 1: [Marks: 20]

Which of the given choices is correct? Write the correct choice number in the following table:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)

(1) If A and B are invertible matrices of same size, then
(i) A + B is invertible (ii) A − B is invertible (iii) A + I is invertible (iv) AB is invertible

(2) If an invertible matrix A satisfies $A^2 = A$, then:
(i) $A = 0$ (ii) $A = I$ (iii) A is singular (iv) $A = -A^T$

(3) The set $\{(\lambda,1), (2\lambda,0)\}$ is linearly dependent in $\mathbb{R}^2$ if:
(i) $\lambda = 0$ (ii) $\lambda = 1$ (iii) $\lambda = 2$ (iv) $\lambda = -1$

(4) Which of the following sets is a basis for $M_2(\mathbb{R})$?
(i) $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}$ (ii) $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\right\}$
(iii) $\left\{\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}$ (iv) $\left\{\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}, \begin{bmatrix}2 & 0\\1 & 3\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}3 & 2\\2 & 4\end{bmatrix}\right\}$

(5) Which of the following sets does not span P_2 (the vector space of polynomials of degree ≤ 2)?
(i) $\{1, x, x^2\}$ (ii) $\{1+2x, x^2\}$ (iii) $\{1+x, x+x^2, 1+x^2\}$ (iv) $\{1, 1+x, 1+x+x^2\}$

(6) Let B and C be two ordered basis of $\mathbb{R}^2$ with the transition matrix $P_{B\to C} = \begin{bmatrix}b & a\\a & 0\end{bmatrix}$. If $w \in \mathbb{R}^2$ with its coordinate vectors $[w]_B = \begin{bmatrix}-1\\1\end{bmatrix}$ and $[w]_c = \begin{bmatrix}1\\2\end{bmatrix}$, then:
(i) a = −2, b = −3 (ii) a = 3, b = 4 (iii) a = 4, b = 3 (iv) a = −2, b = −1

(7) The transition matrix from an ordered basis B to the same ordered basis B must be:
(i) singular (ii) non-diagonal (iii) identity (iv) skew symmetric

(8) Let A = $\begin{bmatrix}x & -x & 3\\0 & x-1 & 1\\0 & 0 & x-1\end{bmatrix}$. The $rank$(A) < 3 if:
(i) $x = -1$ (ii) $x = 1$ (iii) $x = 2$ (iv) $x = 3$

(9) If u and v are vectors in an inner product space such that $||u||^2 = 5, ||v||^2 = 7$ and $< u, v >=$ -2, then the distance $d(u,v)$ between the vector u and v is equal to:
(i) 2 (ii) 4 (iii) $2\sqrt{2}$ (iv) 10

(10) If A is a square matrix of size 4 with det(A) = 3, then $nullity$(A) is equal to:
(i) 4 (ii) 1 (iii) 0 (iv) 3

(11) If a square matrix A of size 2 has eigenvalues 1 and 3, then the eigenvalues of A^3 are:
(i) 1, 3 (ii) 3, 9 (iii) 1, 27 (iv) 3, 27

(12) If $\{(1,1)\}$ is a basis of the eigen space corresponding to the eigenvalue $\lambda = 2$ of the matrix $\begin{bmatrix}-1 & 3\\a & 0\end{bmatrix}$, then a must be:
(i) 1 (ii) 2 (iii) 3 (iv) 0

(13) If $A = \begin{bmatrix} 2 & 1 \\ -1 & m \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ are orthogonal vectors in $M_2(\mathbb{R})$ with respect to the inner product $< A, B >:= trace(AB^T)$, then m must be:

(i) 3 (ii) 2 (iii) 1 (iv) 0

(14) Dimension of the *span*({(1,0,0), (0,2,0), (1,2,0)}) is equal to:

(i) 3 (ii) 2 (iii) 1 (iv) 4

(15) If S = {*u, v, w*} is an orthogonal set of nonzero vectors in $\mathbb{R}^3$, then S must be:

(i) linearly dependent (ii) a subspace of $\mathbb{R}^3$ (iii) a basis of $\mathbb{R}^3$ (iv) an orthonormal subset of $\mathbb{R}^3$

(16) If A is a square matrix such that $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$, then the eigenvalues of A are:

(i) 1 and 2 (ii) 2 and 3 (iii) 1 and 3 (iv) 0 and 2

(17) If T: $\mathbb{R}^2 \to M_2(\mathbb{R})$ is the zero transformation, then *nullity*(T) must be::

(i) 0 (ii) 2 (iii) 4 (iv) *rank*(T)

(18) If T: V→ W is a linear transformation and {*u, v, w*} is a basis of the vector space V, then {T(*u*), T(*v*), T(*w*)} must be:

(i) a basis of the image space *Im*(T) (ii) a spanning set of the image space *Im*(T)
(iii) linearly dependent (iv) a basis of the space *ker*(T)

(19) If the linear system AX = *b* is consistent, then:

(i) $b \in N(A)$ (ii) $b \in row(A^T)$ (iii) $b \in col(A^T)$ (iv) $b \in row(A)$

(20) If a matrix A of size 3×3 has eigenvalues $-3, 0, 3$, then A must be:

(i) invertible and diagonalizable (ii) singular and diagonalizable
(iii) invertible but not diagonalizable (iv) singular but not diagonalizable

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Question 2: [Marks: 2 + 2]

Let A denote the coefficient matrix of the homogeneous system:

$$x + \ \ z = y$$
$$2x + \ \ z = y$$
$$x + 2z = y.$$

a) Find A^{-1} by using the elementary matrix method.
b) Show that the system has no non-trivial solution.

Question 3: [Marks: 2+2+2+2]

Let A be a nonzero matrix of size 3×3 such that $A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then:

a) Show that *nullity*(A) = 2 and *rank*(A) = 1.
b) Show that B = {(0,0,1)), (1,1,1)} is a basis of the null space *N*(A).
c) Apply the Gram-Schmidt algorithm on B to construct an orthonormal basis C of *N*(A).
d) Find the coordinate vector $[(a, b, c)]_C$ for all $(a, b, c) \in N(A)$.

Question 4: [Marks: (1.5+1.5+1)+4]

a) Let $B = \{u_1 = (1, 2), u_2 = (1, 3)\}$ and $C = \{v_1 = 1 + 2x, v_2 = 1 + x^2, v_3 = 1 + x - x^2\}$ be ordered bases of the vector spaces $\mathbb{R}^2$ and P_2, respectively. Let T: $\mathbb{R}^2 \to P_2$ be a linear transformation such that $[T]_B^C = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$ is its associated matrix with respect to the bases B and C. Find $T(1, 0)$, $T(0,1)$ and $T(a, b)$ for all $(a, b) \in \mathbb{R}^2$.

b) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$. Show that the matrix A is diagonalizable, and then compute A^{99}.

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[Solution Key]

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Question 1: [Marks: 20]

Solution: Correct choices:

Version 1:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(iv)	(ii)	(i)	(i)	(ii)	(i)	(iii)	(ii)	(ii)	(iii)
(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
(iii)	(ii)	(iii)	(ii)	(iii)	(ii)	(ii)	(ii)	(ii)	(ii)

Version 2:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(iii)	(i)	(iv)	(iv)	(i)	(iv)	(ii)	(i)	(i)	(ii)
(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
(ii)	(i)	(ii)	(i)	(ii)	(i)	(i)	(i)	(i)	(i)

Version 3:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(ii)	(iv)	(iii)	(iii)	(iv)	(iii)	(i)	(iv)	(iv)	(i)
(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
(i)	(iv)	(i)	(iv)	(i)	(iv)	(iv)	(iv)	(iv)	(iv)

Version 4:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(i)	(iii)	(ii)	(ii)	(iii)	(ii)	(iv)	(iii)	(iii)	(iv)
(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
(iv)	(iii)	(iv)	(iii)	(iv)	(iii)	(iii)	(iii)	(iii)	(iii)

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Question 2: [Marks: 2 + 2]

Solution: a) $[A|I]=\begin{bmatrix}1 & -1 & 1 & : & 1 & 0 & 0\\ 2 & -1 & 1 & : & 0 & 1 & 0\\ 1 & -1 & 2 & : & 0 & 0 & 1\end{bmatrix} \sim \ldots \sim \begin{bmatrix}1 & 0 & 0 & : & -1 & 1 & 0\\ 0 & 1 & 0 & : & -3 & 1 & 1\\ 0 & 0 & 1 & : & -1 & 0 & 1\end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix}-1 & 1 & 0\\ -3 & 1 & 1\\ -1 & 0 & 1\end{bmatrix}$.

b) $A\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix} = A^{-1}\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}-1 & 1 & 0\\ -3 & 1 & 1\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$. So, the system has no non-trivial solution.

Question 3: [Marks: 2 + 2 + 2 + 2]

Solution: a) Since A is a nonzero matrix, $rank(A) \geq 1$. But $A\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} = A\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \Rightarrow nullity(A) \geq 2$. Recall that the matrix A is of size 3×3. Hence, $nullity(A) = 2$ and $rank(A) = 1$.

b) Sinc no vector in B = {(0,0,1)), (1,1,1)} is a scalar multiple of the other, B is linearly independent subset of the null space $N(A)$. Moreover, $dim(N(A)) = 2$ since $nullity(A) = 2$. Hence, B is a basis of the null space $N(A)$.

c) Take $v_1 = (0,0,1)$, and then $v_2 = (1,1,1) - \frac{1}{1}(0,0,1) = (1,1,0)$. Hence, the asked orthonormal basis $C = \{w_1 = \frac{1}{||v_1||}v_1 = (0,0,1), w_2 = \frac{1}{||v_2||}v_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, , 0)\}$.

d) The coordinate vector $[(a,b,c)\,]_C = \begin{bmatrix}< (a,b,c), w_1 >\\ < (a,b,c), w_2 >\end{bmatrix} = \begin{bmatrix}c\\ \frac{1}{\sqrt{2}}(a+b)\end{bmatrix}$, for all $(a,b,c) \in N(A)$.

Question 4: [Marks: (1.5 + 1.5 + 1) + 4]

Solution: a) Since $[T]_B^C = \begin{bmatrix}1 & 2\\ 1 & 0\\ 1 & -1\end{bmatrix}$, we get $[T(u_1)]_C = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$ so that $T(u_1) = 3 + 3x$ and $T(u_2) = 1 + 3x + x^2$.

Now, $T(1,0) = T(3u_1 - 2u_2) = 7 + 3x - 2x^2$ and $T(0,1) = T(-u_1 + u_2) = -2 + x^2$.

Hence, $T(a,b) = T\big(a(1,0) + b(0,1)\big) = 7a - 2b + 3ax + (b - 2a)x^2$ for all $(a,b) \in \mathbb{R}^2$.

b) $A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 1 & -2\end{bmatrix}$ is diagonalizable because it is of size 3×3 and has three different eigenvalues $-1,\ 0,\ 1$. Then, there exists an invertible matrix P such that $P^{-1}AP = \begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$. Hence, $A = P\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}P^{-1}$. Therefore, $A^{99} = P\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}^{99} P^{-1} = P\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}P^{-1} = A$.

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