I. Use mathematical induction to prove that

$$f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}, \quad \forall n \in \mathbb{Z}, n \geq 1$$

(f_n denotes the n-th Fibonacci number).

$P(n): f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$, $n \geq 1$

I $P(1): f_1 = f_2$; $1 = 1$ ✓

II $P(K) \to P(K+1)$

$P(K): f_1 + \cdots + f_{2K-1} = f_{2K}$ ✓

$P(K+1): \underbrace{f_1 + \cdots + f_{2K-1}}_{} + f_{2K+1} \overset{?}{=} f_{2K+2}$

$f_{2K} + f_{2K+1} = f_{2K+2}$ ✓

II. Use the Euclidean Algorithm to find $(204, 444)$ and to write this greatest common divisor as a linear combination of 204 and 444.

$$444 = 204 \times 2 + 36$$

$$204 = 36 \times 5 + 24$$

$$36 = 24 \times 1 + \boxed{12}$$

$$24 = 12 \times 2 + 0$$

$$\Rightarrow (204, 444) = 12$$

$$12 = 36 - 24 = 36 - (204 - 36 \times 5)$$

$$= 36 - 204 + 36 \times 5$$

$$= -204 + 36 \times 6$$

$$= -204 + (444 - 204 \times 2) \times 6$$

$$= -204 + 444 \times 6 - 204 \times 12$$

$$= 204 \times (-13) + 444 \times 6$$

III. Find all solutions of the Diophantine equation

$$2x - 3y - 7z = 10$$

$(3,7)=1 \Rightarrow \exists K \in \mathbb{Z}: \ 3y + 7z = K$

$2X - K = 10$

$(2,1)=1; \ 1 \mid 10$

$2\cdot 1 - 1 = 1 \mid 10$

$2\cdot 10 - 10 = 10 \Rightarrow \begin{cases} X_0 = 10 \\ K_0 = 10 \end{cases}; \quad \begin{cases} X = 10 + n \\ K = 10 + 2n \end{cases} \ , n \in \mathbb{Z}$

$K = 10 + 2n \Rightarrow \ 3y + 7z = 10 + 2n$

$(3,7) = 1$

$7 = 3 \times 2 + \boxed{1}$
$2 = 1 \times 2 + 0$; $\ 1 = 7 - 3 \times 2 \ \Big|(10+2n)$

$3(-20-4n) + 7(10+2n) = 10 + 2n$

$y_0 = -20 - 4n$
$z_0 = 10 + 2n$

$\Rightarrow \begin{cases} y = -20 - 4n + (10+2n)m \\ z = 10 + 2n + (20+4n)m \end{cases}, \quad m \in \mathbb{Z}$

Therefore, $\begin{cases} X = 10 + n \\ y = -20 - 4n + (10+2n)m \\ z = 10 + 2n + (20+4n)m \end{cases}, \quad m, n \in \mathbb{Z}$

IV. Prove that the greatest common divisor of the integers a and b, not both 0, is the least positive integer that is a linear combination of a and b.

V. Show that $\sqrt{5}+\sqrt{2}$ is not a rational number.

If, by contrad, $\sqrt{5}+\sqrt{2} \in \mathbb{Q} \Rightarrow (\sqrt{5}+\sqrt{2})^2 \in \mathbb{Q}$

$5+2+2\sqrt{10} \in \mathbb{Q} \Rightarrow \sqrt{10} \in \mathbb{Q}$; $\sqrt{10} = \frac{a}{b}$; $(a,b)=1$

$10 = \frac{a^2}{b^2} \Rightarrow a^2 = 10b^2 \Rightarrow a \vdots 10 \Rightarrow a = 10k \Rightarrow$
$k \in \mathbb{Z}$

$\Rightarrow 100k^2 = 10b^2 \Rightarrow b^2 = 10k^2 \Rightarrow b \vdots 10 \Rightarrow$

$\Rightarrow b = 10t$, $t \in \mathbb{Z} \Rightarrow (a,b) \neq 1$ contrad

$\Rightarrow \sqrt{5}+\sqrt{2} \notin \mathbb{Q}$