23 Sylow Theorems Tutorial

Exercise 1: Show that conjugacy is an equivalence relation on a group.

Solution: We verify three properties:

- Reflexive: $a = eae^{-1}$ for any $a \in G$.
- Symmetric: If $cac^{-1} = b$, then $a = c^{-1}bc = c^{-1}b(c^{-1})^{-1}$.
- Transitive: If $a = xbx^{-1}$ and $b = ycy^{-1}$, then $a = xycy^{-1}x^{-1} = xyc(xy)^{-1}$.

Exercise 2: If a is a group element, prove that every element in $cl(a)$ has the same order as a.

Solution: For any conjugate gag^{-1}, we have $(gag^{-1})^n = ga^n g^{-1}$. Therefore, $(gag^{-1})^n = e$ if and only if $a^n = e$, proving that $|gag^{-1}| = |a|$.

Exercise 3: Let a be a group element of even order. Prove that a^2 is not in $cl(a)$.

Solution: By Exercise 2, every element in $cl(a)$ has order $|a|$. However, $|a^2| = |a|/2 \neq |a|$ since a has even order. Therefore, $a^2 \notin cl(a)$.

Exercise 6: Show that $cl(a) = \{a\}$ if and only if $a \in Z(G)$.

Solution: $cl(a) = \{a\}$ means that $gag^{-1} = a$ for all $g \in G$. This is equivalent to $ga = ag$ for all $g \in G$, which is precisely the condition that $a \in Z(G)$.

Exercise 15: Suppose that G is a group of order 48. Show that the intersection of any two distinct Sylow 2-subgroups of G has order 8.

Solution: Let H and K be distinct Sylow 2-subgroups of G. By Theorem, $48 \geq |HK| = |H||K|/|H \cap K| = 16 \cdot 16/|H \cap K|$. This simplifies to $|H \cap K| \geq 256/48 > 5$. Since H and K are distinct and $|H \cap K|$ divides 16, we have $|H \cap K| = 8$.

Exercise 19: Suppose that G is a group and $|G| = p^n m$, where p is prime and $p > m$. Prove that a Sylow p-subgroup of G must be normal in G.

Solution: By Theorem 23.5, n_p (the number of Sylow p-subgroups) has the form $1 + kp$ and n_p divides m. If $k \geq 1$, then $1 + kp \geq 1 + p > m$. But $1 + kp$ must divide m. This is a contradiction. Thus $k = 0$ and $n_p = 1$. By the corollary to Theorem 23.5, the unique Sylow p-subgroup is normal.

Exercise 21: Suppose that G is a group of order 168. If G has more than one Sylow 7-subgroup, exactly how many does it have?

Solution: $168 = 8 \cdot 3 \cdot 7$. By Sylow's Third Theorem, the number of Sylow 7-subgroups is $\equiv 1 \pmod 7$ and divides $ 8 $. The divisors of 24 that are $\equiv 1 \pmod 7$ are 1 and 8.

Since there is more than one, there are exactly 8 Sylow 7-subgroups.

Exercise 24: Let G be a noncyclic group of order 21. How many Sylow 3-subgroups does G have?

Solution: By Sylow's Third Theorem, $n_3 \equiv 1 \pmod 3$ and n_3 divides 7. So $n_3 \in \{1, 7\}$. If $n_3 = 1$, then both Sylow subgroups would be normal and unique, making G the internal direct product of cyclic groups of orders 3 and 7. Such a group is cyclic by Theorem 8.2. Since G is noncyclic, we must have $n_3 = 7$.

Exercise 25: Let G be a group of order pq where p and q are distinct primes and $p < q$. Prove that the Sylow q-subgroup is normal in G.

Solution: The number of Sylow q-subgroups has the form $1 + qk$ and divides pq. Since $1 + qk$ must divide p (as it's relatively prime to q), and $1 + qk \geq 1 + q > p$, the only possibility is $1 + qk \leq p$. This forces $k = 0$. Therefore, there is exactly one Sylow q-subgroup, which is normal.

Exercise 44: Suppose that G is a group of order p^n, where p is prime, and G has exactly one subgroup for each divisor of p^n. Show that G is cyclic.

Solution: Since G has a unique subgroup of each order, every subgroup is normal (being the only one of its order). Let $x, y \in G$, then by assumption either $< x > = < y >$ or $< x > \cap < y > = \{e\}$. In either case x and y commute. So G is Abelian. From the fundamental theorem of finite abelian groups, G must be cyclic (as otherwise there would be multiple subgroups of some order).

Exercise 57: Show that a group of order 12 cannot have nine elements of order 2.

Solution: Suppose $|G| = 12 = 2^{23}$ and has nine elements of order 2. By the Sylow Theorems, $n_2 = 3$, i.e. G has three Sylow 2-subgroups whose union contains the identity and the nine elements of order 2. If H and K are both Sylow 2-subgroups, by Theorem 7.2, $|H \cap K| = 2$. Thus the union of the three Sylow 2-subgroups has at most 8 elements of order 2 (3 in H, 2 more in K, and at most 3 more in the third one).