Page 237 – 241 (2) الاختبار النهائي

1
نهائي (2)

For the following mass function :-

(1) find cumulative function $F(x)$

x	-1	0	1	3	Total
$f(x)$	0.1	0.3	0.4	0.2	1
$F(x)$	0.1	0.4	0.8	1	//

$$F(x) = \begin{cases} 0 & x < -1 \\ 0.1 & -1 \leq x < 0 \\ 0.4 & 0 \leq x < 1 \\ 0.8 & 1 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

(2) Find Moment generating function MGF, $M_x(t)$

$$M_x(t) = E(e^{tx})$$

$$= 0.1e^{-t} + 0.3e^{0} + 0.4e^{1t} + 0.2e^{3t}$$

$$= 0.3 + 0.1e^{-t} + 0.4e^{t} + 0.2e^{3t}$$

3) find $M_y(t)$, where $y = 2x+1$

$$= e^{t} M_x(2t)$$

$$= e^{t} M_x(2t)$$

نعوض بدل $t \rightarrow 2t$ in $M_x(t)$

$$= e^{t}\left[0.3 + 0.1e^{-2t} + 0.4e^{2t} + 0.2e^{6t}\right]$$

$$= 0.3e^{t} + 0.1e^{-t} + 0.4e^{3t} + 0.2e^{7t}$$

-ب- If x has r^{th} moment around zero
(1)

$$\boxed{M'_r = E(x^r) = r!}$$

find

① mean: Expected value 2) Variance

① $E(x) = E(x^1) = 1! = 1$
② $E(x^2) = E(x^2) = 2! = 2$

نعوض في معادله $E(x^r)$

$$Var(x) = E(x^2) - [E(x)]^2$$
$$= 2 - 1^2 = 2 - 1 = \boxed{1}$$

-ب- If x has Poisson distribution
(2)
and $f(1) = 2f(2)$

Find (1) $M_x(t)$ (2) $f(3) = ??$

الحل نجد قيمه λ من المعادله المعطاه بالسؤال

Poisson
$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$f(1) = 2f(2)$$

$$\frac{\cancel{e^{-\lambda}}\lambda^1}{1!} = \cancel{2}\frac{\cancel{e^{-\lambda}}\lambda^2}{\cancel{2!}}$$

$$\lambda = \lambda^2 \Rightarrow \frac{\lambda^{\cancel{2}}}{\cancel{\lambda}} = 1$$

$$\Rightarrow \boxed{\lambda = 1}$$

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الاختبار النهائي (3)

تكملة سؤال
- ب -
2

Poisson $\lambda = 1$

$\therefore$ (1) $M_x(t) = e^{\lambda(e^t-1)}$
$= e^{1(e^t-1)} = e^{t-1}$

2) $f(3) = \frac{e^{-\lambda}\lambda^x}{x!}$
$= \frac{e^{-1}1^3}{3!} = \frac{e^{-1}}{6} = \frac{1}{6e}$

Question: السؤال الثاني:

A shooter aims at a specific target if it is known that from previous experience, the Probability of hitting the target is 0.8

Find (1) Probability of <u>not</u> hit the target in $\underbrace{5 \text{ attempt}}_{n=5}$?

$\therefore$ Binomail distribution
$n=5, p=0.8, q=1-p=0.2$

لا يصيب الهدف في 5 محاولات $\Rightarrow$ عدد مرات إصابة الهدف = صفر

$P(X=0) = \binom{n}{x}p^x q^{n-x} = \binom{5}{0}(0.8)^0(0.2)^{5-0}$
$= 1 \times 1 \times (0.2)^5$
$= (0.2)^5$

تكملة السؤال الثاني

(ب) The prob. of ghitting the target for the third time in the 5^{th} attempt

$\Rightarrow$ Negative Binomail $r=3$, $P=0.8$, $q=0.2$

$$P(X=5)=\binom{x-1}{r-1}p^r q^{x-r}$$

$$=\binom{5-1}{3-1}(0.8)^3(0.2)^{5-3}$$

$$=\binom{4}{2}(0.8)^3(0.2)^2 = 0.1229$$

(ج) Prob of hitting the target for the the first time in the fifth ~~...~~ attempt.

$\Rightarrow$ Geometric $P=0.8$, $q=0.2$

$$P(X=5)=pq^{x-1} = 0.8\times(0.2)^{5-1}$$

$$=0.8\times(0.2)^4 = \boxed{0.00128}$$

السؤال الثاني: (ب)

write the distribution (name, parameter, function)

(1) $M_x(t)=\dfrac{e^{3t}-e^{2t}}{t}$

Uniform $(2,3)$ $\quad f(x)=1 \quad 2<x<3$

Continouse

(2) $M_x(t)=(1-2t)^{-8}$;

chi-square distribution with parameter ν

$\frac{\nu}{2}=8 \Rightarrow \boxed{\nu=16}$ $\quad f(x)=\dfrac{1}{2^{\nu/2}\,\Gamma(\nu/2)}\, x^{\frac{\nu}{2}-1}\, e^{-x/\nu}$; $x\geq 0$

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الاختبار النهائي ②

تكملة السؤال الثاني (ب)

(3) $M_X(t) = \left(\frac{4}{4-t}\right)^2$

Gamma dist^n. $M_X(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha$; parameters $\alpha = 2$, $\beta = 4$

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \qquad x \geq 0$$

(4) $M_X(t) = e^{\frac{1}{2}t^2}$

Standard normal $Z \sim N(0,1)$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty$$

السؤال الثالث :

(أ) For X, Y two random variables has the joint moment function $M_{X,Y}(t_1, t_2)$. Prove ×

(ب) If $M_{X,Y}(t_1,t_2) = \frac{1}{1-t_1-t_2+t_1t_2}$; $-\infty < t_1, t_2 < \infty$

find,

(i) $M_X(t_1) = M_{X,Y}(t_1, 0) = \frac{1}{1-t_1-0+\cancel{0t_1}_{\,0}} = \frac{1}{1-t_1}$

نعوض عن $t_2 = 0$

(ii) $M_Y(t_2) = M_{X,Y}(0, t_2) = \frac{1}{1-0-t_2+\cancel{0 \times t_2}_{\,0}} = \frac{1}{1-t_2}$

نعوض عن $t_1 = 0$

(ب) تكملة السؤال الثالث:

(4) Are X, Y independent??

Prove
نثبت أن

$$M_{X,Y}(t_1, t_2) = M_X(t_1) \times M_Y(t_2)$$

$$= \frac{1}{1-t_1} \times \frac{1}{1-t_2}$$

$$= \frac{1}{(1-t_1)(1-t_2)}$$

$$\frac{1}{1-t_1-t_2+t_1 t_2} = \frac{1}{1-t_2-t_1+t_1 t_2}$$

الطرفين متساويين

$\Rightarrow$ X, Y indep.

(ج) If X, Y have the following joint distribution function. (ج)

Find: (1) $f_{X/Y}(1/4) = \frac{f(x,y)}{f_y(y)} = \frac{f(1,2)}{f_y(4)} = \frac{0.25}{0.35}$

$= 5/7$

X	1	2
$f(x)$	$\frac{3}{5}$	$\frac{2}{5}$

y	-3	2	4
$f(y)$	0.35	0.30	0.35

(2) $F_{X/Y}(1/2) = P(X \leq 1 / Y=2) = \frac{f(1,2)}{f_y(2)}$

$= \frac{0.25}{0.30} = 5/6$

Page 237 - 241 الاختبار النهائي (2) ⑦

السؤال الرابع :

If x has a chi_Square distribution with ~~function~~

$$f(x) = k\, x^{2} e^{-x/2} \quad ; x \geq 0$$

(1) find constant k

Chi دالة

$$\boxed{f(x;v) = \frac{1}{2^{\frac{v}{2}}\,\Gamma\left(\frac{v}{2}\right)}\, x^{\frac{v}{2}-1} e^{-x/2} \quad x \geq 0}$$

مع ربط دالة χ^2 مع الدالة المعطاه بالسؤال

نوجد فيه v = ??

$$x^2 = x^{\frac{v}{2}-1}$$

$$\frac{v}{2} - 1 = 2 \Rightarrow \frac{v}{2} = 2+1 = 3$$

$$v = 3 \times 2$$

$$\boxed{v = 6}$$

Now,

$$k = \frac{1}{2^{\frac{v}{2}}\,\Gamma\left(\frac{v}{2}\right)} = \frac{1}{2^{6/2}\,\Gamma\left(\frac{6}{2}\right)}$$

$$= \cancel{\frac{1}{2^3}} = \frac{1}{2^3\,\Gamma(3)} = \frac{1}{2^3 \times 2!} = \boxed{\frac{1}{16}}$$

2) $E(x) = v = 6$

3) $V(x) = 2v = 2 \times 6 = 12$

4) $M_x(t) = (1-2t)^{-v/2} = (1-2t)^{-3}$

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الاختبار النهائي (2)

(8)

تكملة السؤال الرابع (ب)

If X has the following function

$$f_X(x) = \frac{x+1}{15} \qquad x = 0, 1, 2, 3, 4$$

(1) $Y = (X-2)^2$ find $f(y) = ??$

$\cancel{y = x}$

since Y non-to-one

ع نأخذ جدول

السؤال الخامس (أ)

If x, y has the joint prob function

$$f_{y/x}(y/x) = \frac{4-2x-2y}{3-2x} \quad 0 \le x \le 1, \; 0 \le y \le 1$$

and

$$f_x(x) = \frac{3-2x}{2} \quad 0 \le x < 1$$

find,

$$f_{x,y}(x,y) = ?$$

$$f_{y/x}(y/x) = \frac{f(x,y)}{f_x(x)}$$

$$\frac{4-2x-2y}{\cancel{3-2x}} = \frac{f(x,y)}{\frac{\cancel{3-2x}}{2}}$$

$$f(x,y) = \frac{4-2x-2y}{2}$$

$$= \frac{\cancel{2}(2-x-y)}{\cancel{2}}$$

$$f(x,y) = 2-x-y \quad 0 \le x \le 1, \; 0 \le y \le 1$$

2

$$E(y/x) = \int_0^1 y\, f(y/x)\, dy = \int_0^1 y \frac{(4-2x-2y)}{3-2x}\, dy$$

$$= \frac{1}{3-2x}\int_0^1 4y - 2xy - 2y^2\, dy = \frac{1}{3-2x}\left[4\frac{y^2}{2} - 2x\frac{y^2}{2} - 2\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{3-2x}\left(2y^2 - xy^2 - \frac{2}{3}y^3\right]_0^1 = \frac{1}{3-2x}\left[2 - x - \frac{2}{3}\right] = \frac{4/3 - x}{3-2x}$$

$0 \le x \le 1$

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الاختبار الفصلي (2)

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تكملة السؤال الخامس (ب)

If X_1, X_2 indep. find $f_{X_1,X_2}(x_1, x_2) = ??$

الحل

Since X_1, X_2 indep then,

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) * f_{X_2}(x_2)$$

$$= \frac{1}{2} e^{-\frac{x_1}{2}} * \frac{1}{2} e^{-\frac{x_2}{2}}$$

$$= \frac{1}{4} e^{-\frac{1}{2}(x_1 + x_2)} \qquad x_1 \geq 0, \; x_2 \geq 0$$

السؤال الخامس (ج)

(1) Find $f_{Y_1,Y_2}(y_1, y_2) = ?$

$Y_1 = \frac{1}{2}(X_1 - X_2)$ $\qquad Y_2 = X_2$

① $Y_1 = \frac{1}{2}(X_1 - Y_2)$

$2Y_1 = X_1 - Y_2$

$2Y_1 + Y_2 = X_1$

Then,

$g_1^{-1}(y_1, y_2) = 2y_1 + y_2$

$g_2^{-1}(y_1, y_2) = y_2$

② Rang of, $x_1 \geq 0$ $\qquad x_2 \geq 0$

Rang of y_1, y_2 $\qquad y_2 \geq 0$ $\qquad y_2 \geq 0$

3) $$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2\times1 - 1\times0 = 2 - 1 = 1$$

الاختبار الفصلي النهائي (2)

تكملة السؤال الخامس (ج) - (1)

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$\therefore f_{Y_1,Y_2}(y_1,y_2) = |J|\, f_{X_1,X_2}\left[g_1^{-1}(y_1,y_2),\ g_2^{-1}(y_1,y_2)\right]$

$= 1 * f_{X_1,X_2}\left[2y_1+y_2,\ y_2\right]$

$= \frac{1}{4} e^{-\frac{1}{2}(2y_1+y_2+y_2)}$

$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{4} e^{-\frac{1}{2}(2y_1+2y_2)}$

$= \frac{1}{4} e^{-\cancel{\frac{2}{2}}(y_1+y_2)}$

$= \frac{1}{4} e^{-(y_1+y_2)}$

$y_1 \geq 0 \quad , \quad y_2 \geq 0$

---

2) find $f_{Y_1}(y_1) = ??$

$f_{Y_1}(y_1) = \int_0^\infty f(y_1,y_2)\, dy_2 = \int_0^\infty \frac{1}{4} e^{-(y_1+y_2)}\, dy_2 =$

$\frac{1}{4} e^{-y_1} \int_0^\infty e^{-y_2}\, dy_2 = \frac{1}{4} e^{-y_1}\left[-e^{-y_2}\right]_0^\infty = \frac{1}{4} e^{-y_1}(1-0)$

$\therefore f_{Y_1}(y_1) = \frac{1}{4} e^{-y_1} \qquad y_1 \geq 0$

If $\underset{x_1,x_2}{f}(x_1,x_2) = e^{-(x_1+x_2)}$ $0 < x_1, x_2 < \infty$

$Y_1 = X_1 + X_2$ $Y_2 = \dfrac{X_1}{X_1+X_2}$

find $f_{Y_1,Y_2}(y_1,y_2) = ??$

(1) sine $Y_1 = X_1 + X_2 \longrightarrow Y_2 = \dfrac{X_1}{Y_2}$

$Y_1 = Y_1 Y_2 + X_2 \xLeftarrow{\text{then}} X_1 = g^{-1}(y_1) = Y_1 Y_2$

$X_2 = Y_1 - y_1 y_2$

$X_2 = y_1(1-y_2)$

$\therefore g^{-1}(y_1) = Y_1 Y_2$, $g^{-1}(y_2) = y_1(1-y_2)$

(2) $0 < y_1 < \infty$ $0 < y_2 < 1$

(3) $|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} Y_2 & Y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1 y_2 - y_1(1-y_2)$

$= -y_1$

$\therefore f_{Y_1 Y_2}(y_1,y_2) = |J| f_{x_1 x_2}(g^{-1}(y_1), g^{-1}(y_2)) = -y_1 f_{x_1 x_2}(Y_1 Y_2, \underbrace{Y_1(1-Y_2)}_{y_1 - y_1 y_2})$

$= +y_1 e^{-\cancel{y_1 y_2} - y_1 + \cancel{y_1 y_2}}$

$= y_1 e^{-y_1}$

---

(2) $f_{Y_1}(y_1) = \int_0^1 f(y_1,y_2)\, dy_2$

$= \int_0^1 y_1 e^{-y_1} dy_2 = y_1 e^{-y_2}\, y_2 \big|_0^1$

$= y_1 e^{-y_2}$

$f_{Y_2}(y_2) = \int_0^\infty f(y_1,y_2)\, dy_1$

$= \int_0^\infty y_1 e^{-y_1} dy_1 =$ By parts

$= e^{-y}\big|_0^\infty = 1$

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الإختبار النهائي 2

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السؤال السابع :

$$M_{x,y}(t_1, t_2) = \frac{1}{(1-t_1)(1-t_2)} \qquad t_1, t_2 \neq 1$$

(1) $M_x(t_1) = M_{x,y}(t_1, 0) = \frac{1}{1-t_1}$

$M_y(t_2) = M_{x,y}(0, t_2) = \frac{1}{1-t_2}$

(2) X, Y indep if اذا تحقق الشرط

$$M_{x,y}(t_1, t_2) \overset{??}{=} M_x(t_1) * M_y(t_2)$$

$$\frac{1}{(1-t_1)(1-t_2)} \qquad \Big| \qquad \frac{1}{1-t_1} * \frac{1}{1-t_2} = \frac{1}{(1-t_1)(1-t_2)}$$

الطرفين متساويين .

$\therefore$ X, Y are indep

P237-241 الاختبار النزلي (c) : السؤال السابع (14)

(ب)

$$f(x,y) = \frac{x(1+3y^2)}{4} \quad 0<x<2, \quad 0<y<1$$

find

(1) $f_x(x) = \int_0^1 f(x,y)\,dy = \int_0^1 \frac{x(1+3y^2)}{4}\,dy$

$= \frac{x}{4}\int_0^1 1\,dy + \frac{x}{4}\int_0^1 3y^2\,dy$

$= \frac{x}{4}\,y\Big]_0^1 + \frac{x}{4}\,\frac{\cancel{3}y^3}{\cancel{3}}\Big]_0^1 = \frac{x}{4}(1-0) + \frac{x}{4}(1^2-0) = 2\frac{x}{\cancel{4}_2}$

$\boxed{f_x(x) = \frac{x}{2}} \quad 0<x<2$

(2) $f_y(y) = \int_0^2 f(x,y)\,dx = \int_0^2 x\,\frac{(1+3y^2)}{4}\,dx$

$= \frac{1+3y^2}{4}\,\frac{x^2}{2}\Big]_0^2 = \frac{1+3y^2}{8}(2^2-0)$

$\boxed{f_y(y) = \frac{1+3y^2}{2}} \quad 0<y<1$

3) $f_{x/y}(x/y) = \frac{f(x,y)}{f_y(y)} = \frac{\frac{x\cancel{(1+3y^2)}}{\cancel{4}_2}}{\frac{\cancel{(1+3y^2)}}{\cancel{2}_1}} = x/2 \quad 0<x<2$

4) $E(x/y) = \int_0^2 x\,f(x/y)\,dx = \int_0^2 x\cdot\frac{x}{2}\,dx$

$= \int_0^2 \frac{x^2}{2}\,dx = \frac{1}{2}\frac{1}{3}x^3\Big|_0^2 = \boxed{\frac{4}{3}}$

تكملة السؤال السابع

5) $E\,(E(X/Y)] = E\left(\frac{4}{3}\right) = 4/3$

OR

$$E(E(X|Y)) = E(X) = \int_0^2 x\, f_X(x)\, dx$$

$$= \int_0^2 x\, \frac{x}{2}\, dx = \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{6} x^3 \Big|_0^2 = \frac{4}{3}.$$

نهاية الأسئلة فاينال 2

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ملزمة د. الفرايه