

Question 1[4,4] a) Solve the differential equation

$$(y - 4x) \cos x dx + \sin x dy = 0.$$

b) The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time  $t$ . After 3 hours it is observed that 400 bacteria are present. After 100 hours, 2000 bacteria are present. What was the initial number of bacteria.

Question 2[4,4] a) Solve the differential equation:  $xy' - 3y - x^5 y^{\frac{1}{3}} = 0$ .

b) Solve the initial value problem:

$$(y \tan^{-1} x - \frac{y}{1+x^2}) dx - (\tan^{-1} x) dy = 0, \quad y(1) = \frac{\pi}{4}.$$

Question 3[4,4,4] a) Use undetermined coefficients method to find the general solution of the differential equation

$$y'' + 4y = x \sin 2x + 8.$$

b) Find a power series solution to the differential equation

$$y'' + 4x^2 y = 0,$$

about the ordinary point  $x = 0$ .

c) Use variation of parameters method to obtain the general solution of

$$x^2 y'' - xy' = x^2 \ln x, \quad x > 0$$

Question 4[6,6] a) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the  $2\pi$ -periodic odd function defined by:

$$f(x) = x(\pi - x), \quad \text{for } x \in [0, \pi].$$

Sketch the graph of  $f$  on  $[-2\pi, 2\pi]$ . Find the Fourier series of  $f$  and deduce the sum of the numerical series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$ .

(Hint:  $\sin(2n+1)\pi/2 = (-1)^n$ )

b) Let  $f(x) = \begin{cases} |x| & \text{if } |x| \leq 2 \\ 0, & \text{if } |x| > 2 \end{cases}$ . Sketch the graph of  $f$ . Find the

Fourier integral of  $f$  and deduce that  $\int_0^{\infty} \frac{\sin 2\lambda}{\lambda} d\lambda = \int_0^{\infty} \frac{\sin^2 \lambda}{\lambda^2} d\lambda$ .

(Hint:  $\sin^2 \lambda = \frac{1 - \cos(2\lambda)}{2}$ ).