King Saud University College of Since Mathematical Department Differential equations MATH204	جامعة الملك سعود King Saud University	Mid-Term 2 Summer Semester 1440 Time 1H30min Full Mark:25

Question 1 (a): [4] Discuss the existence of unique solution of the following initial value problem

$$\begin{cases} (x-2)y'' + \frac{x}{\sqrt{3-x}}y' + \frac{1}{x^2-4}y = \cos x \\ y(1) = 0, y'(1) = 1 \end{cases}$$

we have $(x-2)$ and $\cos x$ are continues on $\mathbb{R}$

and $\frac{x}{\sqrt{3-x}}$ is continues on $x<3$

and $\frac{1}{x^2-4}$ is continues on $\mathbb{R}-\{2,-2\}$

$a_2(x) \neq 0$ when $x \neq 2$ $\Rightarrow x_0 = 1 \in (-2,2)$

$\therefore$ the initial value problem has unique Solution on $I = (-2,2)$

Question 1 (b): [4] Solve the nonhomogeneous differential equation

$$y'' - 2y' - 3y = e^{2x} + 5\cos 2x$$

we firstly find the Solution y_c of

$y'' - 2y' - 3y = 0$, the charcteristic equation

$m^2 - 2m - 3 = 0 \Rightarrow m_1 = 3, m_2 = -1$

$\Rightarrow y_c = c_1 e^{3x} + c_2 e^{-x}$

Now we will find the y_p Solution

let $f(x) = e^{2x} + 5\cos(2x)$

$f_1(x) = 5\cos(2x) \Rightarrow m = \pm 2i \longrightarrow y_p = A\cos(2x) + B\sin(2x)$

$f_2(x) = e^{2x} \Rightarrow m = 2 \longrightarrow y_p = De^{2x}$

then $\Rightarrow$ $y_p = De^{2x} + A\cos(2x) + B\sin(2x)$

$y'_p = 2De^{2x} - 2A\sin(2x) + 2B\cos(2x)$

$y''_p = 4De^{2x} - 4A\cos(2x) - 4B\sin(2x)$

Sub into $y'' - 2y' - 3y = e^{2x} + 5\cos 2x$

$\Rightarrow D = -\frac{1}{3}, \quad B = \frac{-4}{13}, \quad A = -\frac{7}{13}$

$y = y_c + y_p$

$\therefore$ the general solution

$\Rightarrow y = -\frac{1}{3}e^{2x} - \frac{7}{13}\cos(2x) - \frac{4}{13}\sin(2x) + c_1 e^{3x} + c_2 e^{-x}$

Question 2: [4] show that $y_1 = \sin x$ is a solution of the differential equation

$$y'' + (3\tan x)y' - 2y = 0, \qquad x \in (0, \frac{\pi}{2})$$

then find the second solution and obtain the general solution.

$y_1 = \sin x \Rightarrow y_1' = \cos x \Rightarrow y'' = -\sin x \qquad 0 < x < \frac{\pi}{2}$

$\Rightarrow y_1'' + (3\tan x)y_1' - 2y_1 = -\sin x + 3(\tan x)\cos x - 2\sin x$

$= -3\sin x + 3\sin x = 0$

Thus $y_1 = \sin x$ is a solution of the D.E

$y'' + (3\tan x)y' - 2y = 0, \qquad 0 < x < \frac{\pi}{2}$

thus the second solution y_2 is given by

$y_2 = y_1 \int \frac{e^{\int -P(x)\,dx}}{(y_1)^2}\,dx$, where $P(x) = 3\tan x$

then $e^{\int -3\tan x\,dx} = e^{3\ln(\cos x)} = e^{\ln\cos^3 x} = \cos^3 x$

Thus $y_2 = \sin x \int \frac{\cos^3 x}{\sin^2 x}\,dx = \sin x \int \frac{(1-\sin^2 x)\cos x}{\sin^2 x}\,dx$

by Letting $u = \sin x \Rightarrow du = \cos x\,dx$ then

$I = \int \frac{1-u^2}{u^2}\,du = \int \frac{du}{u^2} - \int du = -\frac{1}{\sin x} - \sin x$

$\therefore y_2 = \sin x\left(\frac{-1}{\sin x} - \sin x\right) = -1 - \sin^2 x.$

therefore the G - Solution is given by

$y = c_1 \sin x - c_2(1 + \sin^2 x)$

$= c_1 \sin x + c_3(1 + \sin^2 x)$

Question 3 (a): [4] show whether the functions

$$f_1(x) = x, \quad f_2(x) = x\ln x$$

are linearly independent or linearly dependent on $(0, \infty)$.

$w(x, f_1(x), f_2(x)) = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x \neq 0$ for all $x > 0$

$\therefore$ The $f_1 = x$ and $f_2 = x\ln x$ are linearly indpendent on $(0, \infty)$.

Question 3 (b): [4] solve the differential equation

$$2x^2y'' - 3xy' - 3y = 0 \quad \text{For } x > 0.$$

Let $y = x^m \Rightarrow y' = mx^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$

Sub into D.E. $2x^2[m(m-1)x^{m-2}] - 3x[mx^{m-1}] - x^m = 0$

$x^m(2m^2 - 2m) - 3mx^m - 3x^m = 0$

$x^m(2m^2 - 2m) - 3mx^m - 3x^m = 0$

$x^m[2m^2 - 5m - 3] = 0$

Since $x^m \neq 0$, the we have $2m^2 - 5m - 3 = 0$

$\Rightarrow m_1 = \frac{1}{2}, \; m_2 = 3$

thus $y(x) = c_1 x^{-\frac{1}{2}} + c_2 x^3$

Question 4: [5] solve the system of the differential equations

$$\begin{cases} 16x'' - y = 0 \\ y'' - 16x = 32t \end{cases}$$

D^2 ⊗ $\begin{cases} 16D^2 x - y = 0 \\ -16x + D^2 y = 32t \end{cases}$

$$16D^4 x - \cancel{D^2 y} = 0$$
$$-16x + \cancel{D^2} y = 32t$$

$\Rightarrow \boxed{16x^{(4)} - 16x = 32t}$

Now we solve this D.E.

$x^{(4)} - x = 0 \quad , x = e^{mt}$

$(m-1)(m+1)(m+i)(m-i) = 0 \Rightarrow m = 1, m = -1, m = i, m = -i$

$\boxed{x_c(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t}$

$x_p(t) = At + B \quad , x_p = -2t$

$\boxed{x(t) = x_c + x_p = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t - 2t}$

Now we solve for (y)

$y = 16x''$, then $x' = c_1 e^{t} - c_2 e^{-t} - c_3 \sin t + c_4 \cos t - 2$

$x'' = c_1 e^{t} + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$

$\boxed{y(t) = 16c_1 e^{t} + 16c_2 e^{-t} - 16c_3 \cos t - 16c_4 \sin t}$