

Question 1 [4,4] a) Find the largest interval for which the following initial value problem has a unique solution

$$\begin{cases} \sqrt{16-x^2}y'' + y' \ln(x+1) + y \cos x = e^{\sqrt{3-x}} \\ y(0) = 2, y'(0) = 0. \end{cases}$$

b) Find a linear differential equation with constant coefficients whose fundamental set of solutions consists of

$$e^{-x}, xe^{-x}, e^x \cos 2x, e^x \sin 2x.$$

Question 2 [4,3]. a) Verify that $y_1 = x+1$ is a solution of the differential equation

$$(1 - 2x - x^2)y'' + 2(x+1)y' - 2y = 0, \text{ for } |x+1| < \sqrt{2}$$

and then obtain its general solution.

b) Determine either the functions

$$f_1(x) = \cos 2x, \quad f_2(x) = 3, \quad f_3(x) = \cos^2 x,$$

are linearly independent or linearly dependent on $(-\infty, \infty)$.

Question 3 [5] Find the general solution of the differential equation

$$x^3y'' - 2xy = 6 \ln x, \quad x > 0.$$

Question 4 [5] Solve the following linear system of differential equations.

$$\begin{cases} x' - x = y - e^t \\ y' = -x - y + e^t \end{cases} \text{ such that } x(0) = y(0) = 1.$$