

Question 1[4,4]

a) Find the largest interval for which the following initial value problem admits a unique solution

$$\begin{cases} (x^3 - x)y'' + \ln(x + 5)y' + y = \cos x \\ y(-\frac{3}{2}) = 0, \quad y'(-\frac{3}{2}) = 1. \end{cases}$$

b) Show that the functions

$$f_1(x) = \ln x, \quad f_2(x) = \ln(2x), \quad f_3(x) = \ln(3x)$$

are linearly dependent on $(0, +\infty)$.

Question 2[4,5]

a) Solve the differential equation

$$y^{(3)} + y' = e^x + 1.$$

b) Find the general of the solution of differential equation

$$x^2 y'' + xy' - y = x^2 + 1, \quad x > 0,$$

where $y_1 = x$ is the solution of the homogeneous equation.

Question 3 [4,4]

a) Determine a linear differential equation with constant coefficients having the solutions

$$y_1 = 5, \quad y_2 = 6e^{-x} \cos(2x), \quad y_3 = e^{-x} \sin(2x), \quad y_4 = 3x, \quad y_5 = 7x^2.$$

b) Solve the differential equation

$$y'' + 4y = \sec(2x), \quad -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}.$$