King Saud University ..... **Date:** 14/10/2026
College of Science..... **Time Allowed:** 90 Minutes
Department of Mathematics

Midterm Exam1: S1-1448-2026
Math 204(Differential Equations)

Total Marks: 25

Question 1:(4,4)

(a) Consider the initial value problem

$$\begin{cases} y' = \dfrac{\ln(9-x^2-y^2)}{x\sqrt{x^2+y^2-1}} \\ y(2) = 1. \end{cases}$$

Find the largest region on which the initial value problem has a unique solution and sketch this region.

(b) Solve the initial value problem

$$\begin{cases} (x^2+2x+2)(y+2)y' = (x+1)(y^2-1),\ y \neq \pm 1, \\ y(0) = 0. \end{cases}$$

Question 2:(4,4)

(a) Solve the initial value problem

$$\begin{cases} (x+1)y' + (x+2)y = 2xe^{-x},\ x > -1 \\ y(0) = 1. \end{cases}$$

(b) Solve the differential equation

$$xydx + (2x^2+3y^2-20)dy = 0,\ x > 0,\ y > 0.$$

Question 3:(4,5)

(a) Find the general solution for the following differential equation

$$\frac{x}{y^{\frac{2}{3}}}y' + y^{\frac{1}{3}} = x^3,\ x > 0,\ y > 0.$$

(b) The population 1000 of a town grows by 2% in a year. If the rate of growth of population is proportional to the population at that instant. What will be the population after 5 years?

***** End of Exam *****

Midterm Exam1: S1-1448-2026

Math 204(Differential Equations)(Exam solution and marks)

Total Marks: 25

Question 1:

(a) We denote by (P) the I.V.P considered.

Let $f(x,y) = \dfrac{\ln(9-x^2-y^2)}{x\sqrt{x^2+y^2-1}}$.

For f to be defined, we need

$$9-x^2-y^2>0,\ x^2+y^2-1>0 \quad \text{and} \quad x\neq 0.$$

Hence

$$1<x^2+y^2<9 \quad \text{and} \quad x\neq 0.$$

Let

$$\begin{aligned} R &= \{(x,y)\in\mathbb{R}^2:\ 1<x^2+y^2<9 \quad \text{and} \quad x\neq 0\} \\ &= R_1\cup R_2, \end{aligned}$$

where

$$R_1=\{(x,y)\in R:\ x>0\},\ R_2=\{(x,y)\in R:\ x<0\}.$$

R_1 and R_2 are two connected regions and $R=R_1\cup R_2$ isn't connected.

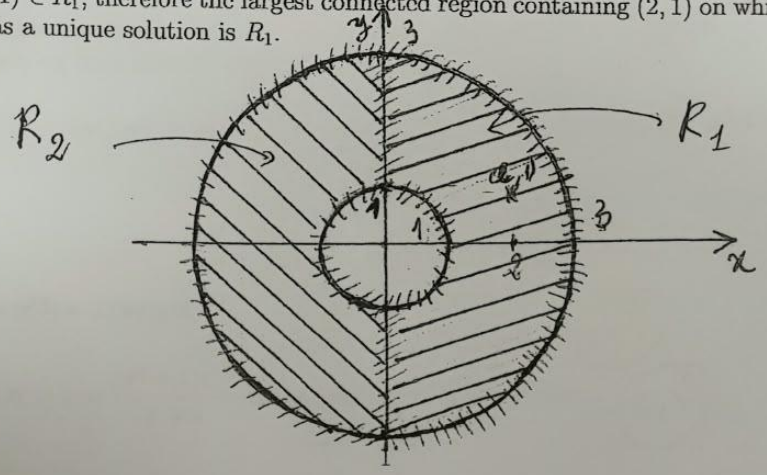
We have:

- f is continuous on R
- f is differentiable w. r. to y on R and

$$\frac{\partial f}{\partial y}(x,y)=\frac{-2y}{(9-x^2-y^2)x\sqrt{x^2+y^2-1}}+\frac{\ln(9-x^2-y^2)}{x}\cdot(-\frac{1}{2})(2y)(x^2+y^2-1)^{-3/2}$$

is continuous on R.

As $(2,1)\in R_1$, therefore the largest connected region containing $(2,1)$ on which the I.V.P. (P) has a unique solution is R_1.



(b) We denote by $(E1)$ the equation considered.
Since $y^2-1=(y-1)(y+1)$, the equation is separable

$$\frac{y+2}{(y-1)(y+1)}dy=\frac{1}{2}\frac{2x+2}{x^2+2x+2}dx.$$

Using

$$\frac{y+2}{(y-1)(y+1)}=\frac{3}{2(y-1)}-\frac{1}{2(y+1)}$$

we obtain

$$\frac{3}{2}\ln|y-1|-\frac{1}{2}\ln|y+1|=\frac{1}{2}\ln(x^2+2x+2)+C.$$

From $y(0)=0$, we obtain

$$C=-\frac{1}{2}\ln(2).$$

Thus

$$\frac{3}{2}\ln|y-1|-\frac{1}{2}\ln|y+1|=\frac{1}{2}\ln(x^2+2x+2)-\frac{1}{2}\ln(2)$$

and

$$3\ln|y-1|-\ln|y+1|=\ln(\frac{x^2+2x+2}{2})$$

is the solution of equation $(E1)$.

Question 2:

(a) We denote by $(E2)$ the equation considered.
Given

$$y'+\frac{x+2}{x+1}y=\frac{2xe^{-x}}{x+1}.$$

Using the fact $\frac{x+2}{x+1}=1+\frac{1}{x+1}$, we get

$$y'+(1+\frac{1}{x+1})y=\frac{2xe^{-x}}{x+1}. \qquad (eq1)$$

Put $P(x)=1+\frac{1}{x+1}$.
I.F. of $(eq1)$ is

$$e^{\int P(x)dx}=e^{\int 1+\frac{1}{x+1}dx}=e^{x+\ln(x+1)+c}=e^c(x+1)e^x.$$

We choose $c=0$ and $\mu(x)=(x+1)e^x$ is an integrating factor of $(eq1)$.
Multiplying both sides of $(eq1)$ by $\mu(x)$, yields

$$(\mu(x)y)'=\mu(x)\frac{2xe^{-x}}{x+1}=2x \Rightarrow \mu(x)y=\int 2xdx+C=x^2+C$$
$$\Rightarrow y=\frac{x^2+C}{\mu(x)}$$
$$\Rightarrow y=\frac{(x^2+C)e^{-x}}{x+1}.$$

Given $y(0)=1$, we get

$$1=0+C\Rightarrow C=1.$$

Therefore, $y=\frac{(x^2+1)e^{-x}}{x+1}$ is the general solution of $(E2)$.

(b) We denote by $(E3)$ the equation considered.
With the identification

$$M(x,y) = xy \text{ and } N(x,y) = 2x^2 + 3y^2 - 20$$

we get

$$M_y = x \neq 4x = N_x.$$

The function defined by

$$\frac{N_x - M_y}{N} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y} := g(y)$$

is a function only in the variable y.
So the integrating factor is

$$e^{\int g(y)dy} = e^{3\ln(y)+c} = e^c e^{\ln(y^3)} = e^c y^3.$$

We choose $c = 0$ and $\mu(y) = y^3$ is an I. F. of $(E3)$.
Multiplying $(E3)$ by $\mu(y)$, we obtain an exact differential equation

$$xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0.$$

Hence, there is a differentiable function $F = F(x,y)$ (w. r. to x and y), such that

$$\begin{cases} \dfrac{\partial F}{\partial x} = xy^4 & (eq2) \\ \dfrac{\partial F}{\partial y} = 2x^2y^3 + 3y^5 - 20y^3 & (eq3) \end{cases}$$

We have

$$(eq2) \Rightarrow F = \int xy^4 dx + h(y) = \frac{1}{2}x^2y^4 + h(y)$$

F and $(y \mapsto \frac{1}{2}x^2y^4)$ are diff. w. r. to y, then h is diff. w. r. to y and

$$\begin{aligned} (eq3) &\Rightarrow 2x^2y^3 + 3y^5 - 20y^3 = \frac{\partial F}{\partial y} = 2x^2y^3 + h'(y) \\ &\Rightarrow h'(y) = 3y^5 - 20y^3 \\ &\Rightarrow h(y) = \int 3y^5 - 20y^3 dy + C \\ &\Rightarrow h(y) = \frac{1}{2}y^6 - 5y^4 + C. \end{aligned}$$

Then, we obtain

$$F = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 + C.$$

Then, the general solution of $(E3)$ is

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 + C = 0.$$