King Saud University,
College of Sciences
Mathematical Department.

Mid-Term Exam/S1/2025
Full Mark:25. Time 1H30mn
08/10/2025

Question 1. [4,5] **a)** Determine and sketch the largest local region of the $xy-$plane for which the initial value problem

$$\begin{cases} (y-x)\sqrt{y}\frac{dy}{dx} = \sqrt{x+1} \\ y(0) = 4, \end{cases}$$

has a unique solution.

b) Find the orthogonal trajectories for the one parameter family of curves

$$y = \tan^{-1}(cx).$$

Question 2. [4, 4]. **a)** Solve the initial value problem

$$\begin{cases} (2y\sin x\cos x + y^2\sin x)dx + (\sin^2 x - 2y\cos x)dy = 0 \\ y(0) = 3 \end{cases}$$

b) Obtain the general solution of the differential equation

$$(x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy.$$

Question 3. [4, 4]. **a)** Solve the initial value problem

$$\begin{cases} (2xy^2 + 3x^2y + 1 + \frac{2}{y})dx + (3x^2y + 2x^3 + \frac{x+3}{y})dy = 0, \ y > 0 \\ y(0) = 1. \end{cases}$$

b) Solve the differential equation

$$x(x^2+y^2)(ydx - xdy) + y^6 dy = 0.$$

(**Hint:** put $ydx - xdy = y^2 d\left(\frac{x}{y}\right)$, $y \neq 0$).

Q2

Consider the initial value problem

$$(y-x)\sqrt{y}\,y' = \sqrt{x+1}, \qquad y(0) = 4.$$

Determine and sketch the largest region containing the point $(0,4)$ on which the Picard–Lindelöf theorem guarantees a unique solution.

Solution

First write the equation in standard form

$$y' = f(x,y) \;=\; \frac{\sqrt{x+1}}{(y-x)\sqrt{y}}.$$

For f (and $\partial f/\partial y$) to be continuous we need the following restrictions:

1. The expression $\sqrt{x+1}$ requires $x+1>0$ (for an open neighborhood we take $x>-1$).
2. The expression $\sqrt{y}$ requires $y>0$.
3. The denominator $(y-x)\sqrt{y}$ must be nonzero, hence $y-x\neq 0$ (i.e. avoid the line $y=x$).

Thus the natural domain for f is

$$D = \{(x,y)\in\mathbb{R}^2:\ x\geq -1,\ y>0,\ y\neq x\}.$$

f and $\partial f/\partial y$ are continuous. The point $(0,4)$ satisfies $x=0>-1$, $y=4>0$, and $y-x=4\neq 0$. It lies in the connected component of the set

$$\{(x,y): x>-1,\ y>0,\ y\neq x\}$$

$$\boxed{R = \{(x,y)\in\mathbb{R}^2:\ x>-1,\ y>0,\ y>x\}.}$$

$$\frac{(6x+1)u'}{3} + 3x^2 + 2u = 0 \quad \text{(is Linear)}$$

(Q6) Find the family of solution of the D.E

$$x(x^2+y^2)(y\,dx - x\,dy) + y^6\,dy = 0$$

, $y \neq 0$ on some interval I

Hint: $y\,dx - x\,dy = y^2 d\left(\frac{x}{y}\right)$

$x = uy$

$$y^2 x(x^2+y^2)\, d\left(\frac{x}{y}\right) + y^6\,dy = 0$$

$$uy^3(u^2y^2+y^2)\,d(u) + y^6\,dy = 0$$

$$uy^5(u^2+1)\,du + y^6\,dy = 0$$

$$u(u^2+1)\,du + y\,dy = 0$$

$$\int (u^3+u)\,du + \int y\,dy = 0$$

$$\frac{1}{4}u^4 + \frac{u^2}{2} + \frac{y^2}{2} = C$$

$$\frac{1}{4}\left(\frac{x^4}{y^4}\right) + \frac{1}{2}\left(\frac{x}{y}\right)^2 + \frac{y^2}{2} = C$$

$$\frac{x^4}{y^4} + \frac{2x^2}{y^2} + 2y^2 = C$$

Hint

$d\left(\frac{x}{y}\right) = \frac{yd}{}$

Q7

$$\boxed{\sin^{-1}(y/x) = \ln x + c}$$

(Q2) Find the general solution of the D. E.

$$(x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy.$$

Solution

$$(x+2)^2 y' = 5 - 4y(x+2) \qquad x > -2$$

$$y' + \frac{4}{x+2}y = \frac{5}{(x+2)^2}$$

$$\mu(x) = e^{\int \frac{4}{(x+2)}dx} = e^{\ln(x+2)^4} = (x+2)^4$$

$$y\,\mu(x) = y(x+2)^4 = \int \frac{5}{(x+2)^2}(x+2)^4\,dx$$

$$y(x+2)^4 = 5\int (x+2)^2\,dx = \frac{5}{3}(x+2)^3 + c$$

$$\boxed{y = \frac{5}{3}\frac{1}{(x+2)} + c\,(x+2)^{-4}}$$

Solution. The given family obeys

$$\frac{\tan y}{x} = c,$$

which on differentiation gives the following differential equation of the family

$$\frac{dy}{dx} = \frac{1}{x}\left(\sin y \cos y\right).$$

The differential equation of the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{x}{\sin y \cos y},$$

which is a variable separable equation with solution

$$\sin^2 y + x^2 = c_1$$

Solution Proposition for M204

Exercise Solve the initial value problem

$$(2y\sin x\cos x + y^2 \sin x)dx + (\sin^2 x - 2y\cos x)dy = 0,\ y(0) = 3$$

Solution

The D.E can be written as: $M(x,y)dx + N(x,y)dy = 0,\ y(0) = 3$

With

$$M(x,y) = 2y\sin x\cos x + y^2 \sin x \text{ and } N(x,y) = \sin^2 x - 2y\cos x$$

Since $M_y(x,y) = 2\sin x\cos x + 2y\sin x = N_x(x,y)$, the DE is exact.

So there exists a function $F(x,y)$ such that

$$\begin{cases} F_x(x,y) = 2y\sin x\cos x + y^2 \sin x \\ F_y(x,y) = \sin^2 x - 2y\cos x \end{cases}$$

Integrating the second equation with respect to y, we obtain

$$F(x,y) = y\sin^2 x - y^2 \cos x + \phi(x)$$

Hence

$$F_x(x,y) = 2y\sin x\cos x + y^2 \sin x + \phi'(x) = 2y\sin x\cos x + y^2 \sin x$$

That is $\phi'(x) = 0$. So $\phi(x) = c_1$.

The general solution is given by $y\sin^2 x - y^2 \cos x = c$

From the Initial condition we obtain $c = -9$.

The required solution is given by $y\sin^2 x - y^2 \cos x + 9 = 0$.

Exercise Solve the initial value problem

$$(2xy^2+3x^2y+1+\frac{2}{y})dx+(3x^2y+2x^3+\frac{x+3}{y})dy=0,\ y(0)=1$$

Solution

The D.E can be written as: $M(x,y)dx+N(x,y)dy=0,\ y(0)=1$

With

$$M(x,y)=2xy^2+3x^2y+1+\frac{2}{y} \text{ and } N(x,y)=3x^2y+2x^3+\frac{x+3}{y}$$

We have $M_y(x,y)=4xy+3x^2-\dfrac{2}{y^2}$ and $N_x(x,y)=6xy+6x^2+\dfrac{1}{y}$

Since $M_y(x,y)\neq N_x(x,y)$, the D.E is not exact.

Note that $\dfrac{N_x(x,y)-M_y(x,y)}{M(x,y)}=\dfrac{1}{y}$

So $\mu(y)=e^{\int\frac{1}{y}dy}=e^{\ln y}=y$ is an integration factor

Hence we have the exact differential equation as

$$(2xy^3+3x^2y^2+y+2)dx+(3x^2y^2+2x^3y+x+3)dy=0,\ y(0)=1$$

Therefore there exists a function $F(x,y)$ such that

$$\begin{cases} F_x(x,y)=2xy^3+3x^2y^2+y+2 \\ F_y(x,y)=3x^2y^2+2x^3y+x+3 \end{cases}$$

Integrating the first equation with respect to x, we obtain

$$F(x,y) = x^2y^3 + x^3y^2 + yx + 2x + \phi(y)$$

Hence

$$F_y(x,y) = 3x^2y^2 + 2x^3y + x + \phi'(y) = 3x^2y^2 + 2x^3y + x + 3$$

That is $\phi'(y) = 3$. So $\phi(y) = 3y + c_1$.

The general solution is given by

$$F(x,y) = x^2y^3 + x^3y^2 + yx + 2x + 3y = c$$

From the Initial condition we obtain $c = 3$.

The required solution is given by

$$x^2y^3 + x^3y^2 + yx + 2x + 3y - 3 = 0.$$