King Saud University, Department of Mathematics
Math 204 (3H), 40/40, Final Exam24/8/39

Question 1[4,4] **a)** Solve the differential equation

$$(y-4x)\cos x dx + \sin x dy = 0.$$

b) The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 100 hours, 2000 bacteria are present. What was the initial number of bacteria.

Question 2[4,4] **a)** Solve the differential equation: $xy' - 3y - x^5 y^{\frac{1}{3}} = 0$.

b) Solve the initial value problem:

$$(y\tan^{-1}x - \frac{y}{1+x^2})dx - (\tan^{-1}x)dy = 0, \quad y(1) = \frac{\pi}{4}.$$

Question 3[4,4,4] **a)** Use undetermined coefficients method to find the general solution of the differential equation

$$y'' + 4y = x\sin 2x + 8.$$

b) Find a power series solution to the differential equation

$$y'' + 4x^2 y = 0,$$

about the ordinary point $x = 0$.

c) Use variation of parameters method to obtain the general solution of

$$x^2 y'' - xy' = x^2 \ln x, \quad x > 0$$

Question 4[6,6] **a)** Let $f : \mathbb{R} \to \mathbb{R}$ be the $2\pi-$periodic odd function defined by:

$$f(x) = x(\pi - x), \quad \text{for } x \in [0, \pi].$$

Sketch the graph of f on $[-2\pi, 2\pi]$. Find the Fourier series of f and deduce the sum of the numerical series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$.

(Hint:$\sin(2n+1)\pi/2 = (-1)^n$)

b) Let $f(x) = \begin{cases} |x| & \text{if} \quad |x| \leq 2 \\ 0, & \text{if} \quad |x| > 2 \end{cases}$ .Sketch the graph of f. Find the Fourier integral of f and deduce that $\int_0^{\infty} \frac{\sin 2\lambda}{\lambda} d\lambda = \int_0^{\infty} \frac{\sin^2 \lambda}{\lambda^2} d\lambda$.

(Hint: $\sin^2 \lambda = \frac{1-\cos(2\lambda)}{2}$).

1% The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 100 hours 2000 bacteria are present. What was the initial number of bacteria?

Q, b

Dr. Mostafa

Solution

Let $P = P(t)$ be bacteria population at time and P_0 the initial number.

$$\frac{dP}{dt} = kP \Rightarrow P = P_0 e^{kt} \quad ①$$

$$P(3) = 400 \Rightarrow 400 = P_0 e^{3k} \quad (1)$$

$$P(10) = 2000 \Rightarrow 2000 = P_0 e^{10k} \quad (2)$$

$$(1) \Rightarrow e^{3k} = \left(400 / P_0\right) \Rightarrow e^{k} = \left(\frac{400}{P_0}\right) \quad ①$$

$$(2) \Rightarrow 2000 = P_0 e^{10k}$$

$$= P_0 \left(\frac{400}{P_0}\right)^{10/3}$$

②

$$\frac{2000}{400^{10/3}} = P_0^{-7/3}$$

$$P_0 = \left(\frac{2000}{100^{10/3}}\right)^{-3/7} \simeq 201$$

Q_2 a

Solution

Rewrite the differential Eq.

$$y' = \frac{3}{x}y + x^4 y^{\frac{1}{3}}$$

$$y' - \frac{3}{x}y = x^4 y^{\frac{1}{3}} \qquad \div \quad y^{\frac{1}{3}}$$

$$\frac{y'}{y^{\frac{1}{3}}} - \frac{3}{x} y^{\frac{2}{3}} = x^4$$

$$u = y^{\frac{2}{3}} \quad , \; u' = \frac{2}{3} y^{-\frac{1}{3}} y' \qquad 1$$

$$\frac{3}{2}u' - \frac{3}{x}u = x^4 \Rightarrow u' - \frac{2}{x}u = ?$$

$$\mu = e^{-\int \frac{2}{x}\,dx} = e^{-\ln x^2} = e^{\ln x^{-2}}$$

$$\mu = \frac{1}{x^2} \qquad 1$$

$$u = x^2 \int \frac{2}{3} x^4 \cdot \frac{1}{x^2}\,dx + c x^2$$

$$u = x^2 \left(\frac{2}{9}x^3 + c \right) \qquad 2$$

$$y(x) = \left(c x^2 + \frac{2}{9} x^5 \right)^{\frac{3}{2}}$$

Q_3 (a)

Dr. Masoud

$y'' + 4y = x \sin 2x + 8$

Ch. Eq.: $m^2 + 4 = 0 \Rightarrow m = \pm 2i$ 1

$y_h = C_1 \cos 2x + C_2 \sin 2x$

$f_1(x) = x \sin 2x \longrightarrow y_{p_1} = (Ax+B)x \sin 2x + (Cx+D)x \cos 2x$

$f_2(x) = 8 \longrightarrow y_{p_2} = E$. 1

$y_p = y_{p_1} + y_{p_2}$, Upon substitution in the DE 1

we get $A = 0$, $B = \frac{1}{16}$, $C = -\frac{1}{8}$, $D = 0$, $E = 2$

$y_g = C_1 \cos 2x + C_2 \sin 2x + \frac{x}{16} \sin 2x - \frac{1}{8} x^2 \cos 2x + 2$ 1

$C_3 b$ **Method of Power Series**

(1) Find a series solution to $y'' + 4x^2 y = 0$ about $x = 0$.

~~The initial conditions are $y(0) = \alpha$, $y'(0) = \beta$.~~

Solution Let $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Dr Mansoor

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 4x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} 4 a_n x^{n+2} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=4}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} 4 a_n x^{n+2} = 0 \quad (1)$$

Let $p = n - 4 \Rightarrow n = p + 4$

$$2a_2 + 6a_3 x + \sum_{n=0}^{\infty} [(n+4)(n+3) a_{n+4} + 4a_n] x^{n+2} = 0$$

$\therefore 2a_2 = 0 \Rightarrow a_2 = 0$, $6a_3 = 0$

and $(n+4)(n+3) a_{n+4} + 4a_n = 0$

$$a_{n+4} = \frac{4a_n}{(n+4)(n+3)} \quad (2)$$

$a_4 = -\frac{1}{3} a_0$, $a_5 = -\frac{1}{5} a_1$, $a_6 = 0$, $a_7 = 0$, $a_8 = \frac{1}{42} a_0$ (1)

$$y(x) = a_0\left(1 - \frac{x^4}{3} + \frac{x^8}{42} \cdots\right) + a_1\left(x - \frac{x^5}{5} + \frac{x^9}{90} \cdots\right)$$

Apply the initial conditions $y(0) = a_0 = \alpha$, $y'(0) = a_1 = \beta$

Thus the final solution is

$$y(x) = \alpha\left(1 - \frac{x^4}{3} + \frac{x^8}{42} \cdots\right) + \beta\left(x - \frac{x^5}{5} + \frac{x^9}{90} \cdots\right)$$ [1]

Q_3 c) : $x^2y'' - xy' = x^2 \ln x$

Ch Eq : $m^2 - 2m = 0 \Rightarrow m_1 = 0, \; m_2 = 2$ 1

$$y_{gh} = C_1 + C_2 x^2$$

$$y_p = C_1(x) y_1 + C_2(x) y_2 = C_1(x) + C_2(x) x^2$$

$$\begin{cases} C_1'(x) + C_2'(x) x^2 = 0 \\ 2x\, C_2'(x) = \ln x \end{cases}$$

$$D = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & x^2 \\ \ln x & 2x \end{vmatrix}}{2x} = \frac{-x^2 \ln x}{2x} = -\frac{1}{2} x \ln x$$

$$C_1(x) = -\frac{1}{2}\left[\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{dx}{x}\right] = -\frac{x^2}{4}\ln x + \frac{x^2}{8}$$ 1

$$C_2'(x) = \frac{\ln x}{2x} \Rightarrow C_2(x) = \frac{1}{2}\int \ln x \, d(\ln x)$$ 1

$$= \frac{1}{4} \ln^2 x$$

$$\boxed{y_g = C_1 + C_2 x^2 + \frac{x^2}{8} - \frac{x^2 \ln x}{4} + \frac{x^2}{4}\ln^2 x}$$ 1

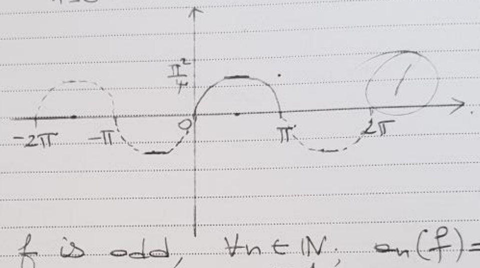
Fourier Series.

Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic odd function defined by:
$f(x) = x(\pi - x)$ if $x \in [0, \pi]$.

1) Sketch the graph of f on $[-2\pi, 2\pi]$
2) Find the Fourier series of f.
3) Deduce the value of the series $\sum_{n=0} \frac{(-1)^n}{(2n+1)^3}$ (Hint: $\sin\left[\frac{(2n+1)\pi}{2}\right] = (-1)^n$)

Solution.

1).



2) Since f is odd, $\forall n \in \mathbb{N}, n \geq 1$, $a_n(f) = 0$

$$b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx \, dx$$

$$= \frac{2}{\pi}\left[\left[x(\pi - x)\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}(\pi - 2x)\cos nx \, dx\right]$$

3

$$= \frac{2}{\pi n}\left(\left[(\pi - 2x)\frac{\sin nx}{n}\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi}\sin nx \, dx\right)$$

$$= \frac{4}{\pi n^3}\left(1 - (-1)^n\right).$$

(2)

Hence $b_{2n}(f) = 0$

$$b_{2n+1}(f) = \frac{8}{(2n+1)^3\pi}.$$

Since f is continuous on $\mathbb{R}$ (C^1 piecewise), by Dirichlet's theorem

$\forall x \in \mathbb{R}$: $$f(x) = \sum_{n\geq 1} b_n(f) \sin nx$$

$$= \sum_{n\geq 0} \frac{8}{(2n+1)^3\pi} \sin(2n+1)x$$

3) From 2)

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n\geq 0} \frac{\sin (2n+1)\frac{\pi}{2}}{(2n+1)^3}$$

2

Therefore

$$\frac{\pi^2}{4} = \frac{8}{\pi} \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3}$$

and so, $$\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$$

Fourier integral. (Q4 b)

Consider the function f defined by.

$$f(x)=\begin{cases} |x| & ; |x|\leq 2 \\ 0 & ; |x|>2 \end{cases}$$

1) sketch the graph of f

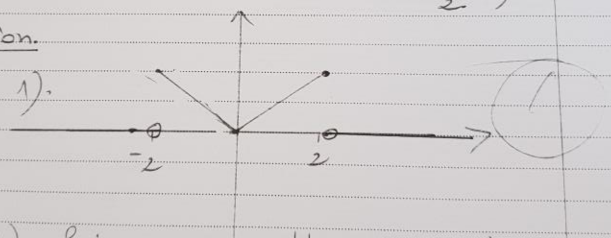
2) Find the Fourier integral of f

3) Deduce that, $\displaystyle\int_0^{\infty} \frac{\sin 2\lambda}{\lambda}\,d\lambda = \int_0^{\infty} \frac{\sin^2\lambda}{\lambda^2}\,d\lambda$

"(Hint: recall that $\sin^2\lambda = \frac{1-\cos 2\lambda}{2}$)".

Solution.

1).



2) f is even then $B(\lambda)=0$

$$A(\lambda)=\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt = 2\int_0^{2} t\cos\lambda t\,dt =$$

$$= 2\left(\left[t\,\frac{\sin\lambda t}{\lambda}\right]_0^2 - \int_0^2 \frac{\sin\lambda t}{\lambda}\,dt\right)$$

$$= 2\left(2\frac{\sin 2\lambda}{\lambda} + \left[\frac{\cos \lambda t}{\lambda^2} \right]_0^2 \right)$$

$$= 2\left(\frac{2\sin 2\lambda}{\lambda} + \frac{\cos 2\lambda}{\lambda^2} - \frac{1}{\lambda^2} \right)$$

$$= \frac{2 \cdot (\cos 2\lambda + 2\lambda \sin 2\lambda - 1)}{\lambda^2}$$

3

So, $\forall x \neq \pm 2$

$$f(x) = \frac{1}{\pi} \int_0^\infty A(\lambda) \cos \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{(\cos 2\lambda + 2\lambda \sin 2\lambda - 1)}{\lambda^2} \cos \lambda x$$

(3) If $x = 0$, then

$$\int_0^\infty \frac{\cos 2\lambda + 2\lambda \sin 2\lambda - 1}{\lambda^2} \, d\lambda = 0$$

2

Then $\displaystyle\int_0^\infty \frac{1 - \cos 2\lambda}{\lambda^2} \, d\lambda = 2\int_0^\infty \frac{\sin 2}{\lambda}$