KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
TIME: 3H, FULL MARKS: 40, SII /22/08/1438, MATH 204

Question 1. **[4,5]** **a)** Show that $\mu(x,y) = x^{-2}y^{-3}$ is an integrating factor for the differential equation

$$(x^3y - y^2)dx - (x^4 + xy)dy = 0, \quad x > 0,\ y > 0.$$

b) A mass of radio active material was left in a lab. After 1 year the mass decreased by 4% and after 10 years it was found that 80 gram of the material was left. Find the initial mass and the half life of this material.

Question 2. **a)** **[4,3].** Solve the initial value problem

$$\frac{dy}{dx} = \frac{y^3 + 2x^2y}{xy^2 + x^3}, \quad y(1) = 1.$$

b) Find a linear differential equation that has the general solution

$$y = c_1e^x + c_2xe^x + c_3\cos 2x + c_4\sin 2x$$

Question 3. **a)** **[4,4].** Use the undetermined coefficients method to solve the second order differential equation

$$y'' - 2y' + 2y = 4x - \cos x$$

b) Solve the differential equation

$$x^2y'' - 2y = 2\ln x, \quad x > 0.$$

Question 4 **[5,5].** a) Find the first six terms in a power series expansion about the ordinary point $x_0 = 0$ for a general solution to the equation

$$y'' = xy.$$

b) Use the method of elimination to solve the system of differential equations

$$\begin{cases} x' = -3x + 2y \\ y' = -3x + 4y \end{cases}$$

Question 5. **[6].** Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function defined by

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

Sketch the graph of f over $(-3\pi, 3\pi)$. Find the Fourier series of the function f and deduce the value of the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

$M = \frac{F \times 5}{8}$

$M_2 = \max\{M_1, M\}$

Answer Sheet

Q. a) Multiply the DE by $\mu(x,y) = x^{-2}y^{-3}$, we get

$$\underbrace{(xy^{-2} - x^{-2}y^{-1})}_{M}dx - \underbrace{(x^2y^{-3} + x^{-1}y^{-2})}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = -2xy^{-3} + x^{-2}y^{-2}, \quad \frac{\partial N}{\partial x} = -2xy^{-3} + x^{-2}y^{-2} \quad (1)$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{(The DE is exact)} \Rightarrow \exists F(x,y)$$

$$\begin{cases} \frac{\partial F}{\partial x} = xy^{-2} - x^{-2}y^{-1} \longrightarrow (1) \\ \frac{\partial F}{\partial y} = -x^2y^{-3} - x^{-1}y^{-2} \longrightarrow (2) \end{cases} \quad (1)$$

From (1), $F(x,y) = \frac{x^2}{2}y^{-2} + x^{-1}y^{-1} + \alpha(y) \longrightarrow (3)$ (1)

From (3) $\frac{\partial F}{\partial y} = -x^2y^{-3} - x^{-1}y^{-2} + \alpha'(y) \longrightarrow (4)$

From (2) and (4), we have $\alpha'(y) = 0 \Rightarrow \alpha(y) = C_1$ (1)

Hence $\frac{x^2}{2}y^{-2} + x^{-1}y^{-1} = C$ is the sol of the DE.

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b) Let $A(t)$ be the mass present at time t,
and A_0 is the initial mass.

$A(1) = A_0 - \frac{4}{100}A_0 = 0.96A_0 \longrightarrow ①$ (1)

$A(10) = 80 \longrightarrow ②$ (1)

Since $A(t) = A_0e^{kt}$

$① \Rightarrow 0.96A_0 = A_0e^k \Rightarrow k = \ln 0.96 \simeq -0.041$ (1)

Thus $A = A_0e^{-0.041t}$

$② \Rightarrow 80 = A_0e^{-0.41} \Rightarrow A_0 = 80e^{0.41} \simeq 120.55.$ (1)

Half-life: $\frac{1}{2} = e^{-0.041t} \Rightarrow t = \frac{\ln 0.5}{-0.041} \simeq 16.91$ years.

(1)

Q_2 a) : $y' = \frac{y^3 + 2x^2 y}{xy^2 + x^3}$ the numerator and the denomentor have the same degree, so the DE is homogeneous.

$\frac{dy}{dx} = y' = \frac{\left(\frac{y}{x}\right)^3 + 2\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1}$, let $u = \frac{y}{x} \Rightarrow y' = xu' + u$. (1)

Hence $xu' + u = \frac{u^3 + 2u}{u^2 + 1} \Rightarrow x\frac{du}{dx} = \frac{u}{u^2+1} \Rightarrow \frac{(u^2+1)}{u}du = \frac{dx}{x}$

$\Rightarrow \int \left(u + \frac{1}{u}\right)du = \ln|x| + c \Rightarrow \frac{u^2}{2} + \ln|u| = \ln|x| + c$ (1)

$\Rightarrow \frac{y^2}{2x^2} + \ln\left|\frac{y}{x}\right| - \ln|x| = c \Rightarrow \frac{y^2}{2x^2} + \ln\left(\frac{|y|}{x^2}\right) = c$ (1)

$\Rightarrow \operatorname{Exp}\left\{\frac{y^2}{2x^2}\right\} \cdot \frac{|y|}{x^2} = c^* \Rightarrow \exp\left\{\frac{y^2}{2x^2}\right\}\frac{y}{x^2} = \pm c^* = C_1$

Since $y(1) = 1$, $C_1 = \sqrt{e}$

Hence $\exp\left\{\frac{y^2}{2x^2}\right\}\frac{y}{x^2} = \sqrt{e}$. (1)

Q_2. b) $m_1 = 1$, $m_2 = 1$, $m_3 = 2i$, $m_4 = -2i$. (1)

The characteristic equation is $(m-1)^2(m-2i)(m+2i) = 0$

After multiplication, we get: $m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$ (1)

Thus the DE is $y^{(4)} - 2y^{(3)} + 5y'' - 8y' + 4y = 0$ (1)

Q3 a) $y'' - 2y' + 2y = 4x - \cos x$

$y_g = y_{gh} + y_p$

$y_{gh} = ?$ Ch Eq: $m^2 - 2m + 2 = 0 \Rightarrow m_1 = 1 + i,\ m_2 = 1 - i$

$y_{gh} = e^{x}(C_1 \cos x + C_2 \sin x)$. (1)

$y_p = Ax + B + C\cos x + D\sin x$ (1)

$y_p' = A - C\sin x + D\cos x$

$y_p'' = -C\cos x - D\sin x$

By substitution in the DE:

$-C\cos x - D\sin x - 2A + 2C\sin x - 2D\cos x + 2Ax + 2B$

$+ 2C\cos x + 2D\sin x = 4x - \cos x$

$\Leftrightarrow \cos x(C - 2D) + \sin x(D + 2C) - 2A + 2B + 2Ax = 4x - \cos x$

Equating, we get $A = 2,\ B = 2,\ C = -\frac{1}{5},\ D = \frac{2}{5}$

Hence $y_g = e^{x}(C_1 \cos x + C_2 \sin x) + 2x + 2 - \frac{1}{5}\cos x + \frac{2}{5}\sin x$

(2)

1. Q_3 b) ; $x^2y'' - 2y = 2\ln x$, $x>0$

$y_g = y_{h} + y_p$

$x^2y'' - 2y = 0$: the Ch Eq: $m^2 - m - 2 = 0 \Rightarrow m_1 = -1, m_2 = 2$

(1)

$\Rightarrow y_{h} = C_1 x^{-1} + C_2 x^2$,

$y_p = C_1(x)x^{-1} + C_2(x)x^2$, where

$$\begin{cases} C_1'(x)x^{-1} + C_2'(x)x^2 = 0 \\ -C_1'(x)x^{-2} + 2C_2'(x)\cdot x = \frac{2\ln x}{x^2} \end{cases}$$

$$W = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3$$

$$C_1'(x) = \begin{vmatrix} 0 & x^2 \\ \frac{2\ln x}{x^2} & 2x \end{vmatrix} / 3 = -\frac{2\ln x}{3}$$ (1)

$$\Rightarrow C_1(x) = -\frac{2}{3}(x\ln x - x)$$

$$C_2'(x) = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{2\ln x}{x^2} \end{vmatrix} / 3 = \frac{2}{3}\frac{\ln x}{x^3}$$ (1)

$$\Rightarrow C_2(x) = \frac{2}{3}\int \frac{\ln x}{x^3}dx = -\frac{1}{3x^2}\ln x - \frac{1}{6x^2}$$

Hence $y_g = C_1x^{-1} + C_2x^2 - \frac{2}{3}(x\ln x - x)x^{-1} - \left(\frac{1}{3x^2}\ln x + \frac{1}{6x^2}\right)x^2$

$$\boxed{y_g = C_1 x^{-1} + C_2x^2 + \frac{1}{2} - \ln x}$$ (1)

Q_4 a) $y'' - xy = 0$, $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_1^{\infty} n a_n x^{n-1}$, $y'' = \sum_2^{\infty} n(n-1)a_n x^{n-2}$

Hence $\sum_2^{\infty} n(n-1)a_n x^{n-2} - \sum_0^{\infty} a_n x^{n+1} = 0$

$\Leftrightarrow \sum_0^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_1^{\infty} a_{n-1}x^n = 0$ (1)

$\Leftrightarrow 2a_2 + \sum_1^{\infty}[(n+2)(n+1)a_{n+2} - a_{n-1}]x^n = 0$

$\Rightarrow a_2 = 0$, $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$, $n \geq 1$ (2)

$n=1$ $a_3 = \frac{a_0}{6}$

$n=2$: $a_4 = \frac{a_1}{12}$, $n=3$; $a_5 = \frac{a_2}{20} = 0$, $n=4$ $a_6 = \frac{a_0}{180}$

Hence $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$ (2)

$$= a_0\left[1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right] + a_1\left[x + \frac{x^4}{12} + \cdots\right]$$

Q4 b) operator form $\begin{cases} (D+3)[x] - 2y = 0 \longrightarrow (1) \\ 3x + (D-4)[y] = 0 \longrightarrow (2) \end{cases}$ (1)

To eliminate for example x, we apply $(D+3)$ to (2) and multiply (1) by -3, we obtain:

$$(D^2 - D - 6)[y] = 0 \Rightarrow y'' - y' - 6y = 0 \quad (1)$$

$$m^2 - m - 6 = 0 \Rightarrow m_1 = 3, \; m_2 = -2$$

$$y(t) = c_1 e^{3t} + c_2 e^{-2t} \; (1), \quad y' = 3c_1 e^{3t} - 2c_2 e^{-2t}$$

From (2), we have $x(t) = -\frac{y'}{3} + \frac{4y}{3}$

$$= -\frac{1}{3}\left(3c_1 e^{3t} - 2c_2 e^{-2t}\right) + \frac{4}{3}\left(c_1 e^{3t} + c_2 e^{-2t}\right)$$

$$\Rightarrow x(t) = \frac{c_1}{3} e^{3t} + 2c_2 e^{-2t} \quad (2)$$

Q_5: Since f, and f' are piecewise continuous on $(-\pi, \pi)$ and since $f(x+2\pi) = f(x)$, then f has the Fourier series (1)

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} (x+\pi)\,dx = \frac{1}{2\pi}(x+\pi)^2 \Big|_{-\pi}^{0} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (x+\pi)\cos nx\,dx = \frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin nx}{n}\,dx\right]$$

$$= -\frac{1}{\pi n}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} = \frac{1}{\pi n^2}\left(1-(-1)^n\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (x+\pi)\sin nx\,dx = \frac{1}{\pi}\left[-(x+\pi)\frac{\cos nx}{n}\Big|_{-\pi}^{0} + \int_{-\pi}^{0}\frac{\cos nx}{n}\,dx\right]$$

$$= -\frac{1}{n} - \frac{\sin nx}{n^2}\Big|_{-\pi}^{0} = -\frac{1}{n}$$ (2)

Hence
$$f(x) \simeq \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)}{(2n+1)^2} - \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Let $x = 0$

$$f(0) \simeq \frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$ (1.)

$$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

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