King Saud University, Mathematics Department
Math 204. Time: 3H, Full Marks: 40, 19/05/2016
Final Exam

Question 1[4,4]. **a)** Determine the region in the $xy-$plane for which the following differential equation

$$(y-x)y' = x+y,$$

would have a unique solution through the point $(1,-3)$.

b) By using separation of variables, solve the differential equation

$$(e^y+1)^2 e^x dy + (e^x+1)e^y dx = 0.$$

Question 2. [5,5] a) By finding an appropriate integrating factor, solve the following differential equation

$$(y^2+xy^3)dx + (5y^2 - xy + y^3 \sin y)dy = 0, \quad x \neq 0, \ y \neq 0.$$

b) Bacteria in a culture increased from 400 to 1600 in 3 hours. Assuming that the rate of increase is directly proportional to the population. Find the number of bacteria at the end of 6 hours.

Question 3. [4,4] a) Use undetermined coefficients method to solve the differential equation

$$y'' - 2y' + y = 2e^x - 3e^{-x}.$$

b) Find the general solution of

$$x^2y'' - 3xy' + 4y = \sqrt{x}, \quad x > 0,$$

if $y_1 = x^2$ is a solution for the homogeneous equation.

Question 4. [4] Find the power series solution near the ordinary point $x_0 = 0$ of the differential equation

$$y'' - (x+1)y' - y = 0.$$

Question 5. [5,5] a) Sketch the given $2\pi-$periodic function f and obtain its Fourier series

$$f(x) = \begin{cases} \cos x, & \text{if } x \in (0,\pi) \\ -\cos x, & \text{if } x \in (-\pi, 0) \end{cases}$$

Deduce the value of the series $\sum_{n=1}^{\infty} \frac{n \sin 2n}{4n^2-1}$.

Hint: $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$,
$\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$.

b) Find the Fourier integral of the function

$$g(x) = \begin{cases} \cos x, & \text{if } |x| \leq \pi \\ 0, & \text{if } |x| > \pi, \end{cases}$$

and deduce that $\int_0^{\infty} \frac{\lambda \sin \lambda\pi}{1-\lambda^2} d\lambda = \pi/2$.

P1

Answer sheet. Final Exam Math 204

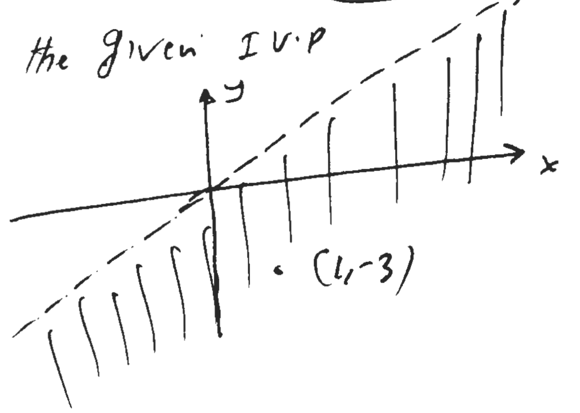
A 1) a) $f(x,y) = \frac{y+x}{y-x}$ is continuous on the set

$D = \{(x,y) \in \mathbb{R}^2 : y \neq x\}$ 1

$\frac{\partial f}{\partial y} = \frac{-2x}{(y-x)^2}$ is continuous on D. 1

Since $(1,-3) \in D^* = \{(x,y) \in \mathbb{R}^2 : y < x\}$ 2

where $f, \frac{\partial f}{\partial y}$ are continuous, then the given I.V.P has a unique solution on D^*.



b) $(e^y+1)^2 e^x dy + (e^x+1) e^y dx = 0$

This is a separable equation. Separation of variables gives

$(e^y+1)^2 e^{-y} dy = -(e^x+1) e^{-x} dx$ 1

$\Leftrightarrow \int (e^y+1)^2 e^{-y} dy = -\int (e^x+1) e^{-x} dx$

let $e^{-y} = u \Rightarrow e^{-y} dy = -du$, $e^{-x} = v \Rightarrow e^{-x} dx = -dv$

Then we get $-\int \left(\frac{1}{u}+1\right)^2 du = +\int \left(\frac{1}{v}+1\right) dv$ 2

$\Leftrightarrow \frac{1}{u} - 2\ln|u| - u = \ln|v| + v + C$

$\Leftrightarrow e^y + 2y - e^{-y} = -x + e^{-x} + C$ 1

(P2)

A2 a) $(y^2+xy^3)dx+(5y^2-xy+y^3\sin y)dy=0$ (*)

Here $\dfrac{N_x-M_y}{M}=-\dfrac{3}{y}$ (2)

$\Rightarrow \mu(y)=e^{-3\int\frac{dy}{y}}=\dfrac{1}{y^3}$

Multiplication of (*) by $\mu(y)=y^{-3}$ gives

$\left(\dfrac{1}{y}+x\right)dx+\left(\dfrac{5}{y}-\dfrac{x}{y^2}+\sin y\right)dy=0$ (**)

(**) is exact $\Rightarrow \exists\ F(x,y)$ such that

$$\begin{cases}\dfrac{\partial F}{\partial x}=\dfrac{1}{y}+x & \longrightarrow (1)\\[2ex] \dfrac{\partial F}{\partial y}=\dfrac{5}{y}-\dfrac{x}{y^2}+\sin y & \longrightarrow (2)\end{cases}$$

From (1), $F(x,y)=\dfrac{x}{y}+\dfrac{x^2}{2}+h(y)\rightarrow(3)$ (1)

(2) and (3) implies: $-\dfrac{x}{y^2}+h'(y)=\dfrac{5}{y}-\dfrac{x}{y^2}+\sin y$

Hence $h(y)=+5\ln|y|-\cos y$ (2)

The Sol of the DE is: $\dfrac{x}{y}+\dfrac{x^2}{2}+5\ln|y|-\cos y=C$

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b) The mathematical model $\begin{cases}\dfrac{dP}{dt}=kP\\ P(0)=P_0=400\end{cases}$ (1)

$\dfrac{dP}{P}=k\,dt \Leftrightarrow P(t)=Ce^{kt}$, $P(0)=C=400$ (2)

$\Rightarrow P(t)=400e^{kt}$

$P(3)=400e^{3k}=1600 \Rightarrow e^{3k}=4\Rightarrow k=\dfrac{2\ln 2}{3}$

Hence $P(t)=400e^{(2\ln 2/3)t}$ (2)

$P(6)=400e^{\ln 16}=400\times 16=6400.$

P3

A3 a) $y'' - 2y' + y = 2e^x - 3e^{-x}$

$$y_g = y_{gh} + y_p$$

$y'' - 2y' + y = 0 \Rightarrow m^2 - 2m + 1 = 0 \Rightarrow m_1 = m_2 = 1$ (1)

$$y_{gh} = C_1 e^x + C_2 x e^x.$$

(1)

$$y_p = Ax^2e^x + Be^{-x}$$

$$y_p' = 2Axe^x + Ax^2e^x - Be^{-x}$$

$$y_p'' = 2Ae^x + 4Axe^x + Ax^2e^x + Be^{-x}$$

By substitution in the DE, we have

$$2Ae^x + 4Be^{-x} = 2e^x - 3e^{-x}$$

$$\Leftrightarrow \begin{array}{l} 2A = 2 \Rightarrow A = 1 \\ 4B = -3 \Rightarrow B = -\frac{3}{4} \end{array}$$

Hence $y_p = x^2e^x - \frac{3}{4}e^{-x}$ (2)

$$y_g = C_1 e^x + C_2 x e^x + x^2e^x - \frac{3}{4}e^{-x}$$

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b) Let $y = y_1 u = x^2 u$ (1)

$y' = 2xu + x^2u'$, $y'' = 2u + 4xu' + x^2u''$

The DE becomes:

$$x^4u'' + 4x^3u' + 2x^2u - 6x^2u - 3x^3u' + 4x^2u = \sqrt{x}$$

$\Leftrightarrow x^4u'' + x^3u' = \sqrt{x}$, Let $u' = v \Rightarrow u'' = v'$

Thus: $x^4v' + x^3v = \sqrt{x}$ (LE)

standard form: $v' + \frac{v}{x} = x^{-\frac{7}{2}}$ (1)

$$\mu(x) = e^{\ln x} = x$$

(P4)

Then $\frac{d}{dx}(xv) = x^{-5/2} \Longrightarrow xv = -\frac{2}{3}x^{-3/2} + C_1$

$$\Longrightarrow v = -\frac{2}{3}x^{-5/2} + \frac{C_1}{x}$$

recalling: $v = u' = -\frac{2}{3}x^{-5/2} + \frac{C_1}{x}$

$$\Longrightarrow u = \frac{4}{9}x^{-3/2} + C_1 \ln x + C_2$$ (2)

So $y = x^2\left[\frac{4}{9}x^{-3/2} + C_1 \ln x + C_2\right]$

$$y = \frac{4}{9}\sqrt{x} + C_1 x^2 \ln x + C_2 x^2$$

A4: $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

The DE becomes:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - (x+1)\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$ (1)

$$\Leftrightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow (2a_2 \ast a_1 - a_0)x^0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - a_n(n+1)\right]x^n = 0$$

$$\Longrightarrow a_2 = \frac{a_0 + a_1}{2}$$ (1)

$$a_{n+2} = \frac{(n+1)a_{n+1} + a_n(n+1)}{(n+1)(n+2)}, \quad n \geq 1$$ (1)

$\underline{n=1}$ $a_3 = \frac{2a_2 + 2a_1}{6} = \frac{1}{3}\left(\frac{a_0 + a_1}{2}\right) + \frac{a_1}{3} = \frac{a_0}{6} + \frac{a_1}{2}$

$\underline{n=2}$: $a_4 = \frac{3a_3 + 3a_2}{12} = \frac{1}{4}\left(\frac{a_0}{6} + \frac{a_1}{2}\right) + \frac{1}{4}\left(\frac{a_0 + a_1}{2}\right)$

$$= \frac{a_0}{6} + \frac{a_1}{4}$$

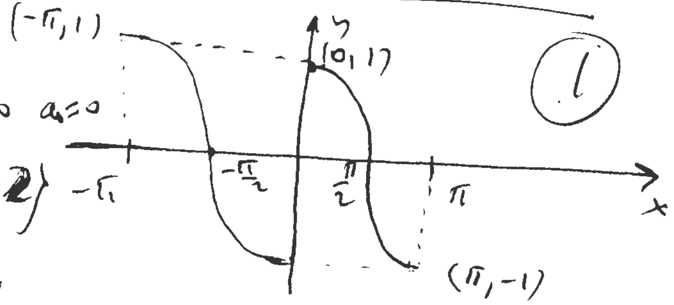
$$y = a_0 + a_1 x + \left(\frac{a_0}{2} + \frac{a_1}{2}\right)x^2 + \left(\frac{a_0}{6} + \frac{a_1}{2}\right)x^3 + \left(\frac{a_0}{6} + \frac{a_1}{4}\right)x^4 + \cdots$$

Sol: (Q5)

$$y = a_0\Big[\underbrace{1+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{6}+\cdots}_{y_1}\Big] + a_1\Big[\underbrace{x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{4}+\cdots}_{y_2}\Big]$$ (1)

Q6 a) f is odd, then $a_n = 0$, $a_0 = 0$

It is continuous on $\mathbb{R}\setminus\{k\pi, k\in\mathbb{Z}\}$ (1)



$$b_n = \frac{2}{\pi}\int_0^\pi \cos x \sin nx\,dx = \frac{1}{\pi}\int_0^\pi \big(\sin(n+1)x + \sin(n-1)x\big)\,dx$$

odd

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^\pi \quad (n\neq 1)$$

$$= \frac{2n}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right], \quad n \neq 1$$ (1)

$$b_1 = \frac{2}{\pi}\int_0^\pi \cos x \sin x\,dx = 0$$

$$f(x) = \sum_{n=2}^{\infty} \frac{2\cdot n}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right]\sin nx$$ (2)

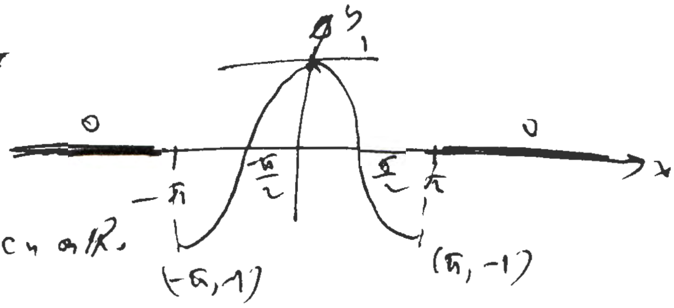
$$= \frac{2}{\pi}\left[\frac{2\cdot\cancel{2}\sin 2x}{2^2-1} + \frac{2\cdot\cancel{4}}{4^2-1}\sin 4x + \cdots + \frac{2\cdot 2n}{(2n)^2-1}\sin 2nx + \cdots\right]$$

$$= \frac{\cancel{8}}{\pi}\sum_{n=1}^{\infty} \frac{2n\sin 2nx}{4n^2-1}$$ (1)

$$f(1) = \cos x = \frac{\cancel{8}}{\pi}\sum_{n=1}^{\infty}\frac{n\sin 2n}{4n^2-1}$$

(1.6)

b) $f(x) = \begin{cases} \cos x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$



$f(-x) = f(x) \Rightarrow f$ is even on $\mathbb{R}$.

$\forall x \neq \pm\pi$: $\quad f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)\,d\lambda$

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)\,dt = 2\int_0^\infty f(t)\cos(\lambda t)\,dt$$

$$= 2\int_0^\pi \cos t\cos(\lambda t)\,dt$$

$$= \int_0^\pi \left(\cos(1+\lambda)t + \cos(\lambda-1)t\right)dt$$

$$= \left[\frac{\sin(1+\lambda)t}{1+\lambda} + \frac{\sin(\lambda-1)t}{\lambda-1}\right]_0^\pi$$

$$= \frac{\lambda\sin\lambda\pi}{1-\lambda^2} \qquad 2$$

~~f(0) = 1~~ Hence

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda\sin(\lambda\pi)\cos(\lambda x)}{1-\lambda^2}\,d\lambda \qquad 1$$

If $x=0$, then

$$f(0) = 1 = \frac{2}{\pi}\int_0^\infty \frac{\lambda\sin(\lambda\pi)}{1-\lambda^2}\,d\lambda$$

$$\Rightarrow \int_0^\infty \frac{\lambda\sin(\lambda\pi)}{1-\lambda^2}\,d\lambda = \pi/2 \qquad 2$$

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