King Saud University, Department of Mathematics
Math 204 (3H), 40/40, Final Exam10/8/36

Question 1[4,4] **a)** Test if the following equation is exact, if it is not, find the appropriate integrating factor and solve it.

$$(x^2y + y^2)dx + (x^3 - 2xy)dy = 0$$

b) Solve the inital value problem

$$\begin{cases} \frac{1}{9y^2}dx - \frac{1}{x(e^{x^2}+3x)}dy = 0 \\ y(1) = 1 \end{cases}$$

Question 2[4,4] **a)** Solve the initial value problem

$$\begin{cases} x^2y' - 2xy - y^3 = 0, \quad x > 0 \\ y(1) = 1. \end{cases}$$

b) Find the family of orthogonal trajectories of the family of curves

$$xye^{x^2} = c.$$

Question 3[4,5,5] **a)** Find the general solution of the differential equation

$$y^{(4)} + y^{(3)} = 1 - e^{-x}.$$

b) Use power series method to solve the homogeneous equation

$$y'' - xy' + xy = 0,$$

about the ordinary point $x = 0$.

c) Use variation of parameters method to obtain the general solution of

$$x^2y'' - xy' + y = 4x\ln x, \quad x > 0$$

Question 4[5,5] **a)** Let f be a periodic function of period 2π given by:

$$f(x) = \pi^2 - x^2 \qquad \text{for } x \in (-\pi, \pi).$$

Find the Fourier series of f and deduce the sum of the numerical series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

b) Find the Fourier integral of the function $f(x) = \begin{cases} C & \text{if} \quad |x| \le 1 \\ 0, & \text{if} \quad |x| > 1 \end{cases}$, where $C \neq 0$.

Deduce the value of the integral $\int_0^{\infty} \frac{\sin\lambda}{\lambda} d\lambda$.

Answer sheet

Final exam 2014

(P1)

Q_1 a) $(x^2y+y^2)dx+(x^3-2xy)dy=0$

$$\frac{\partial M}{\partial y}=x^2+2y \quad , \quad \frac{\partial N}{\partial x}=3x^2-2y$$

So the equation is not exact.

1

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{-2x^2+4y}{x(x^2-2y)}=-\frac{2}{x}.$$

Thus $\mu(x)=e^{\int -\frac{2}{x}dx}=\frac{1}{x^2}$ is an I.F

we get: $\left(y+\frac{y^2}{x^2}\right)dx+\left(x-\frac{2}{x}y\right)dy=0$

$$\frac{\partial}{\partial y}\left(y+\frac{y^2}{x^2}\right)=1+\frac{2y}{x^2}=\frac{\partial}{\partial x}\left(x-\frac{2}{x}y\right)$$

2

we solve
$$\begin{cases}\frac{\partial f}{\partial x}=y+\frac{y^2}{x^2} & (1)\\ \frac{\partial f}{\partial y}=x-\frac{2}{x}y & (2)\end{cases}$$

From (1) $f(x,y)=xy-\frac{y^2}{x}+h(y)$

using (2) $\frac{\partial f}{\partial y}=x-\frac{2y}{x}+h'(y)=x-\frac{2}{x}y$

So $h'(y)=0 \qquad h=\text{constant}$

P_2

the solution is given by

$$xy - \frac{y^2}{x} = c$$ (1)

Q1 b) we have

$$9y^2 dy = x(e^{x^2}+3x)\,dx$$

$$3y^3 = \frac{1}{2}e^{x^2} + x^3 + c$$ (2)

$y(1)=1$ gives

$$2 = \frac{e}{2}+c \quad , \quad c = -\frac{e}{2}+2$$ (2)

$$y = \left(\frac{\frac{1}{2}e^{x^2}+2+x^3-\frac{e}{2}}{3}\right)^{1/3}$$

Q2 a) $x^2y' - 2xy - y^3 = 0$

$$y' - \frac{2}{x}y = \frac{y^3}{x^2}$$ Bernoulli's D.E with $n=3$

$$y^{-3}y' - \frac{2}{x}y^{-2} = \frac{1}{x^2} \quad (*)$$

set $w = y^{-2}$, $w' = -2y^{-3}y'$ (1)

(*) becomes $-\frac{w'}{2} - \frac{2}{x}w = \frac{1}{x^2}$

$$w' + \frac{4}{x}w = -\frac{2}{x^2}$$ linear D.E

(P3)

$$\mu(x) = e^{\int \frac{4}{x} dx} = x^4$$ (2)

$$\mu(x)\, w(x) = \int \mu(x)\left(\frac{-2}{x^2}\right) dx$$

$$= \int -2x^2 \, dx$$

$$x^4 y^{-2} = -\frac{2}{3} x^3 + c$$ (1)

$$y(1) = 1 \Rightarrow c = \frac{5}{3}$$

$$\frac{x^4}{y^2} = \frac{-2}{3} x^3 + \frac{5}{3}.$$

b) $x y e^{x^2} = c$ differentiate

$$(y + x y') e^{x^2} + 2x e^{x^2} x y = 0$$

$$y + x y' = -2x^2 y$$

$$y' = -\frac{(2x^2+1)\, y}{x}$$ (1)

For the orthogonal trajectories we

Solve $y' = \frac{x}{2x^2+1} \frac{1}{y}$ (1) separable

$$\int y \, dy = \int \frac{x}{2x^2+1} dx$$

$$y^2 = \frac{1}{2} \ln(2x^2+1) + 2K$$ a

(P4)

Q3 a) $y^{(4)} + y^{(3)} = 1 - e^{-x}$.

(1)

C.E $m^4 + m^3 = 0 \iff m^3(m+1) = 0$ $\begin{cases} m = 0 \\ \text{or} \\ m = -1 \end{cases}$

$y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$.

The particular solution has the form $y_p = Ax^3 + Bx e^{-x}$ (1)

$y_p^{(4)} + y_p''' = 6A - Be^{-x} = 1 - e^{-x}$ (2)

thus $A = \frac{1}{6}$, $B = 1$.

$y_p = \frac{1}{6}x^3 + x e^{-x}$ (4)

The solution is

$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + \frac{1}{6}x^3 + x e^{-x}$.

b) $y = \sum_0^\infty a_n x^n$, $y' = \sum_1^\infty n a_n x^{n-1}$

$y'' = \sum_2^\infty n(n-1) a_n x^{n-2}$

$y'' - xy' + xy = 0$ leads to.

$$\sum_{0}(n+2)(n+1)a_{n+2}x^{n} - \sum_{1} n a_n x^n + \sum_{1} a_{n-1} x^n$$

$$2a_2 + \sum_{1}^{\infty}\left((n+2)(n+1)a_{n+2} - n a_n + a_{n-1}\right)x^n = 0$$

(1)

So $\begin{cases} 2a_2 = 0 \\ a_{n+2} = \dfrac{n a_n - a_{n-1}}{(n+2)(n+1)} \quad n \geq 1 \end{cases}$

(2)

$$a_3 = \frac{1}{6}(a_1 - a_0)$$

$$a_4 = \frac{2a_2 - a_1}{12} = -\frac{1}{12}a_1$$

$$a_5 = \frac{3a_3 - a_2}{5.4} = \frac{1}{40}(a_1 - a_0)$$

$$a_6 = \frac{4a_4 - a_3}{6.5} = -\frac{(a_1 - \frac{1}{3}a_0)}{60}$$

$$y = \sum a_n x^n$$

$$= a_0 + a_1 x + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$= a_0 + a_1 x + \frac{1}{6}(a_1 - a_0)x^3 - \frac{1}{12}a_1 x^4 + \frac{1}{40}(a_1 - a_0)x^5$$

$$\cancel{+} \frac{1}{60}(a_1 - \tfrac{1}{3}a_0)x^6 + \cdots$$

(2)

$$= a_0\left(1 - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{180}x^6 + \cdots\right)$$

$$+ a_1\left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 - \frac{1}{60}x^6 + \cdots\right)$$

c) we first solve

$$x^2y'' - xy' + y = 0$$

a Cauchy-Euler's equation

its auxiliary equation is

$$m^2 - 2m + 1 = 0 \iff (m-1)^2 = 0 \qquad m = 1$$

1

$$y_c = c_1 x + c_2 x\ln x \quad , \quad y_1 = x \ , \ y_2 = x\ln x$$

y_p has the form $y_p = u_1 y_1 + u_2 y_2$

we solve $\begin{cases} u_1' x + u_2' x\ln x = 0 \\ u_1' + u_2'(1+\ln x) = 4\dfrac{\ln x}{x} \end{cases}$

$$W(x, y_1, y_2) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x \neq 0 \quad \text{for } x > 0$$

2

$$W_1 = \begin{vmatrix} 0 & x\ln x \\ \dfrac{4\ln x}{x} & 1+\ln x \end{vmatrix} = -4(\ln x)^2$$

$$u_1' = -4\frac{(\ln x)^2}{x} \quad \text{so} \quad u_1 = -\frac{4}{3}(\ln x)^3$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \dfrac{4\ln x}{x} \end{vmatrix} = 4\ln x$$

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$$u_2' = \frac{|x|_2}{|x|} = \frac{4\ln x}{x}$$

$$u_2 = 2(\ln x)^2$$

$$y_p = -\frac{4}{3}x(\ln x)^3 + 2x(\ln x)^3$$

$$= \frac{2}{3}x(\ln x)^3$$

2

$$y = y_c + y_p = c_1 x + c_2 x\ln x + \frac{2}{3}x(\ln x)^3.$$

Q4 a) f is even

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \pi^2 - x^2 \, dx = \frac{4\pi^2}{3}$$

1

For $n \geq 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)\cos nx \, dx$$

$$\int_0^{\pi} (\pi^2 - x^2)\cos nx \, dx = \pi^2 \underbrace{\int_0^{\pi} \cos nx \, dx}_{0} - \int_0^{\pi} x^2 \cos nx \, dx$$

$$= -\left\{ \underbrace{\left[\frac{x^2}{n}\sin nx\right]_0^{\pi}}_{0} - \int_0^{\pi} \frac{2x}{n}\sin nx \, dx \right\}$$

$$= \frac{-2\pi(-1)^n}{n^2}$$

2

$$a_n = \frac{4(-1)^{n+1}}{n^2} \quad n \geq 1$$

P2

$$Sf(x) = \frac{2\pi^2}{3} + 4\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$$

at $x = 0$

$$Sf(0) = f(0) = \pi^2 = \frac{2\pi^2}{3} - 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2}$$

So $$\sum_{1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

b) f is even

$$A(\lambda) = c\int_{-1}^{1} \cos \lambda t \, dt = \frac{2c \sin \lambda}{\lambda}$$

2

$$B(\lambda) = 0$$

The Fourier integral is given by

1

$$\frac{1}{\pi}\int_{0}^{+\infty} \frac{2c \sin \lambda}{\lambda} \cos \lambda x \, d\lambda = f(x)$$

for $x = 0$ we get

2

$$\int_{0}^{+\infty} \frac{\sin \lambda}{\lambda} d\lambda = \frac{\pi}{2}.$$