KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
TIME: 3H, FULL MARKS: 40, SII /30/07/1435, MATH 204

Question 1. [4,5] **a)** Solve the following linear differential equation

$$xy' + (1+2x^2)y = x^3e^{-x^2}, \qquad x > 0.$$

b) An adult takes 400 mg of aspirin. Each hour, the amount of aspirin in the body decreases by 25%. How much aspirin will be left after 3 hours, if it is known that the amount of the aspirin in the body decreases at a rate proportional to the amount present in the body at any time t.

Question 2. a) [4,4]. Solve the initial value problem

$$\begin{cases} (y^2 + xy + x^2)dx - x^2dy = 0 \\ y(1) = 1 \end{cases}$$

b) Find the general solution of the Bernoulli equation

$$y' + (\sin x).y = (\sin x).y^2$$

Question 3. a) [4,4]. Use the undetermined coefficients method to solve the second order differential equation

$$y'' - 2y' - 3y = 36e^{5x}$$

b) Show that the solutions: $y_1 = 1$, $y_2 = e^x$, $y_3 = xe^x$ of the differential equation

$$y^{(3)} - 2y'' + y' = 0$$

are linearly independent on $(-\infty, +\infty)$ and deduce its general solution.

Question 4 [5]. Find the first five terms in a power series expansion about the ordinary point $x_0 = 0$ for a general solution to the equation

$$y'' - xy = 0$$

Question 5. a) [5,5]. Let: $f(x) = \pi - |x|, \quad -\pi \le x \le \pi$,
such that $f(x+2\pi) = f(x)$. Find the Fourier series of the function f and deduce that: $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

b) Find the Fourier integral representation for the function

$$f(x) = \begin{cases} 3, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

and deduce that $\int_0^{\infty} \frac{\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}$.

Answer Sheet
Final Exam Math 204

Q1 a) $xy' + (1+2x^2)y = x^3 e^{-x^2}$, $x>0$

$\Leftrightarrow \quad y' + \left(\frac{1}{x} + 2x\right)y = x^2 e^{-x^2} \longrightarrow (*)$

$\mu(x) = e^{\int(\frac{1}{x}+2x)dx} = e^{\ln x} \cdot e^{x^2} = x e^{x^2}$ (2)

Multiply the standard form $(*)$ by $\mu(x)$, we get

$$\frac{d}{dx}\left(x e^{x^2} y\right) = x^3$$ (2)

$$\Rightarrow \quad x e^{x^2} y = \frac{x^4}{4} + c$$

$$\Rightarrow \quad y = \left(\frac{x^3}{4} + \frac{c}{x}\right) e^{-x^2} \quad \#.$$

b) Let $Q(t)$ be the quantity of the Aspirin

$\frac{dQ}{dt} = KQ \Rightarrow \frac{dQ}{Q} = k\,dt \Rightarrow Q(t) = Ce^{kt}$ (1)

$Q(0) = 400 = C \Rightarrow Q(t) = 400 e^{kt}$

$Q(1) = 400e^{k} = 300 \Rightarrow K = \ln\frac{3}{4} =$ (2)

$Q(t) = 400 e^{t \ln\frac{3}{4}} = 400 \left(\frac{3}{4}\right)^t$

$Q(3) = 400 \; \frac{27}{64} \simeq 168.75$ mg (2)

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Q_2 : $(y^2 + xy + x^2)\,dx - x^2\,dy = 0$ (2)

a) $y' = \dfrac{y^2 + xy + x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) + 1 = H\left(\frac{y}{x}\right)$ (1)

The DE is homogeneous.

Let $v = \frac{y}{x} \Rightarrow y' = v + xv'$

Hence $v + xv' = v^2 + v + 1$ (1)

$\Rightarrow xv' = v^2 + 1$ (Sep Eq)

$\Rightarrow \dfrac{dv}{v^2+1} = \dfrac{dx}{x} \Rightarrow \tan^{-1} v = \ln|x| + c$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \ln|x| = c$ (1)

$y(1) = 1 \Rightarrow \tan^{-1}(1) = c = \frac{\pi}{4}$.

Thus $\tan^{-1}\left(\frac{y}{x}\right) - \ln|x| = \frac{\pi}{4}$. (1)

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b) $y' + (\sin x)\,y = (\sin x)\,y^2$ (BE)

$\Rightarrow y^{-2}y' + (\sin x)\,y^{-1} = \sin x$

Let $u = y^{-1} \Rightarrow u' = -y^{-2}y'$ (1)

then $-u' + u\sin x = \sin x$ (LE)

Standard form $u' - u\sin x = -\sin x \longrightarrow (*)$

$\mu(x) = e^{-\int \sin x\,dx} = e^{\cos x}$ (1)

Multiply $(*)$ by $\mu(x)$, we get

$\frac{d}{dx}\left(u\,e^{\cos x}\right) = -\sin x\, e^{\cos x} = \frac{d}{dx}\left(e^{\cos x}\right)$ (2)

$\Rightarrow u\,e^{\cos x} = e^{\cos x} + C$

$\Rightarrow u = 1 + C e^{-\cos x}$

$\Rightarrow y^{-1} = 1 + C e^{-\cos x} \Rightarrow y = \dfrac{1}{1 + C e^{-\cos x}}$

$\underline{Q_3}$ a) $y'' - 2y' - 3y = 36 e^{5x}$

$y_g = y_c + y_p$

Charact Eq: $m^2 - 2m - 3 = 0 \Rightarrow (m+1)(m-3) = 0$, $m_1 = -1$, $m_2 = 3$

$y_c = C_1 e^{-x} + C_2 e^{3x}$ 1

$y_p = A e^{5x}$, $y_p' = 5A e^{5x}$, $y_p'' = 25 A e^{5x}$ 1

$(25A - 10A - 3A) e^{5x} = 36 e^{5x}$

$\Rightarrow A = \dfrac{36}{12} = 3$

$y_p = 3 e^{5x}$ 2

$y_g = C_1 e^{-x} + C_2 e^{3x} + 3 e^{5x}$

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b) $y''' - 2y'' + y' = 0$, $y_1 = 1$, $y_2 = e^x$, $y_3 = x e^x$ 3

$$W[y_1, y_2, y_3](x) = \begin{vmatrix} 1 & e^x & x e^x \\ 0 & e^x & (1+x) e^x \\ 0 & e^x & (2+x) e^x \end{vmatrix} = \begin{vmatrix} e^x & (1+x) e^x \\ e^x & (2+x) e^x \end{vmatrix} = e^{2x} \neq 0$$

$\forall x \in \mathbb{R}$

$\Rightarrow y_1 = 1$, $y_2 = e^x$, $y_3 = x e^x$ are linearly independent on $\mathbb{R} = (-\infty, \infty)$. 1

Hence $y_c = C_1 + C_2 e^x + C_3 x e^x$.

(4)

$y'' - xy = 0$

$\sum_{2} n(n-1) a_n x^{n-2} - \sum_{0}^{\infty} a_n x^{n+1} = 0$

$\sum_{0} (n+2)(n+1) a_{n+2} x^n - \sum_{1} a_{n-1} x^n = 0$ (1)

$2a_2 = 0 \Rightarrow a_2 = 0$

$(n+2)(n+1) a_{n+2} = a_{n-1} \quad \forall n \geq 1$ (Recurrence formula)

$$a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)} \quad n \geq 1$$ (2)

$\underline{n=1}$ $a_3 = \frac{a_0}{6}$

$\underline{n=2}$: $a_4 = \frac{a_1}{12}$

$\underline{n=3}$: $a_5 = 0$

$\underline{n=4}$: $a_6 = \frac{a_3}{30} = \frac{a_0}{180}$ (2)

$y = a_0 + a_1 x + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 + \cdots$

$y = a_0\left[1 + \frac{x^3}{6} + \cdots\right] + a_1\left[x + \frac{x^4}{12} + \cdots\right]$

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Q5: a) $f(x) = \pi - |x|$

$f(-x) = f(x) \quad \forall x \in [-\pi, \pi]$

f is even, thus $b_n = 0$ 1

$$a_0 = \frac{2}{\pi}\int_0^\pi (\pi - x)\,dx = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^\pi = \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos nx\,dx = \frac{2}{\pi}\left[\frac{(\pi - x)\sin nx}{n}\Big|_0^\pi + \int_0^\pi \frac{\sin nx}{n}\,dx\right]$$ 1

$$= -\frac{2}{\pi n^2}\cos nx\Big|_0^\pi = \frac{2}{\pi n^2}\left[1 - (-1)^n\right] = \begin{cases} 0, & n \text{ even} \\ \frac{4}{\pi n^2}, & n \text{ odd}\end{cases}$$

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left(1 - (-1)^n\right)\cos nx$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$ 1

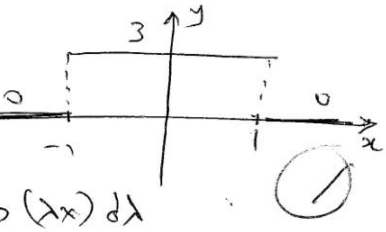
At $x = 0$, the FS converges to $f(0)$, hence

$$f(0) = \pi = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$ 2

b) f is even, $B(\lambda) = 0$,

$A(\lambda) = 2\int_0^\infty f(t)\cos\lambda t\,dt$



and

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)\,d\lambda$$ 1

$$A(\lambda) = 2\int_0^1 3\cos(\lambda t)\,dt = 6\,\frac{\sin(\lambda t)}{\lambda}\Big|_0^1$$ 6

$$= \frac{6\sin\lambda}{\lambda}.$$ 2

Hence $f(x) = \frac{6}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\cos(\lambda x)\,d\lambda$

At $x = 0$, the FI converges to $f(0)$, hence

$$f(0) = 3 = \frac{6}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$$ 2

$$\Longrightarrow \int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{\pi}{2}.$$