KING SAUD UNIVERSITY, DEPARTMENT OF MATHEMATICS

MATH 204. TIME: 3H, FULL MARKS: 40, FINAL EXAM

Question 1. [4,4,5] **a)** Solve the initial value problem

$$\begin{cases} (x + ye^{y/x})dx - xe^{y/x}dy = 0 \\ y(1) = 0 \end{cases}$$

b) Solve the differential equation $y(e^{-2x} + y^2)dx - e^{-2x}dy = 0$.

c) A thermometer is taken from an inside room to the outside, where the air temperature is 5^0F. After 1 minute the temperature reads 55^0F, and after 5 minutes it reads 30^0F. What is the initial temperature of the inside room.

Question 2. [4,5] **a)** If $y_1 = x^{-1}$ is a solution of the differential equation $x^2y'' + xy' - y = 0$, $x > 0$,use reduction of order to solve the differential equation

$$x^2y'' + xy' - y = \ln x, \quad x > 0.$$

b) Find the largest interval for which the following initial value problem admits a unique solution

$$\begin{cases} \frac{y''}{x^2-1} + (\tan x)y = e^x \\ y(0) = 1, \ y'(0) = 0 \end{cases}$$

Question 3. [3,5] **a)** Determine the form of the particular solution of the following differential equation

$$y''' - 3y' + 2y = x^2e^x + 3e^{-x} + \sin 2x.$$

b) Use power series method to find the first four nonzero terms of the solution of the initial value problem

$$\begin{cases} y'' + 3xy' - y = 0 \\ y(0) = 2, \quad y'(0) = 0 \end{cases}$$

Question 4. [5,5] **a)** Consider the $2\pi-$periodic even function defined by

$$f(x) = 1 - \frac{2x}{\pi}, \text{ for } x \in [0, \pi].$$

Sketch the graph of f on $(-3\pi, 3\pi)$, obtain the Fourier series for the function f, and deduce the value of the numerical series $\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)^2}$, and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.

b) Consider the function

$$f(x) = \begin{cases} 1 + x, & -1 \le x \le 0, \\ 1 - x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Sketch the graph of f, find the Fourier integral representation of f, and deduce the value of $\int\limits_0^{\infty} \frac{\sin^2 \lambda}{\lambda^2} d\lambda$.

Answer Sheet.

Q1 a) $(x+ye^{\frac{y}{x}})dx - xe^{\frac{y}{x}}dy = 0$, $y(1)=0$

Divide both sides by x :

$$\left(1+\frac{y}{x}e^{\frac{y}{x}}\right)dx - e^{\frac{y}{x}}dy = 0 \quad (HE)$$

Let $u=\frac{y}{x} \Rightarrow y' = xu'+u$, then (2)

$$y' = xu'+u = \frac{1+ue^{u}}{e^{u}} = e^{-u}+u$$

$$\Rightarrow e^{u}du = \frac{dx}{x}$$

$$\Rightarrow \ln|x| - e^{u} = C$$

$$y(1)=0 \Rightarrow C=-1 \quad (2)$$

Hence $e^{\frac{y}{x}}+1 = \ln|x|$

Q1 b) $y(e^{-2x}+y^2)dx - e^{-2x}dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{y(e^{-2x}+y^2)}{e^{-2x}}$$

$$\Rightarrow y' - y = e^{2x}y^3 \quad (BE)$$

$$\Rightarrow y'y^{-3} - y^{-2} = e^{2x} \quad (2)$$

Let $v = y^{-2} \Rightarrow v' = -2y^{-3}y'$

Hence $v' + 2v = -2e^{2x} \quad (LE)$

$$\mu(x) = e^{2\int dx} = e^{2x}.$$

Then $\frac{d}{dx}\left(e^{2x}v\right) = -2e^{4x}$ (2)

$$\Rightarrow e^{2x}v = -\frac{e^{4x}}{2} + C$$

Hence $v = y^{-2} = -\frac{e^{2x}}{2} + Ce^{-2x}$

c) $\begin{cases} \frac{dT}{dt} = k(T-T_s), \quad T_s = 5 \\ T(0) = T_0 \end{cases}$

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$T(t) = 5 + Ce^{kt}$.

$T(0) = T_0 = 5 + C \Rightarrow C = T_0 - 5$

Hence $T(t) = 5 + (T_0 - 5)e^{kt}$

$$\begin{aligned} T(1) &= 55 = 5 + (T_0 - 5)e^{k} \\ T(5) &= 30 = 5 + (T_0 - 5)e^{5k} \end{aligned} \Rightarrow \begin{cases} (T_0 - 5)e^{k} = 50 \\ (T_0 - 5)e^{5k} = 25 \end{cases}$$

Hence $e^{4k} = \frac{1}{2} \Rightarrow k = -\frac{\ln 2}{4}$.

Then $T_0 = 5 + 50e^{\frac{\ln 2}{4}} \simeq 64.46\,°F$.

Q2 If $y_1 = x^{-1}$ is a solution of the differential equation

a)

$$x^2y'' + xy' - y = 0, \quad x > 0$$

use reduction of order to solve the differential equation

$$x^2y'' + xy' - y = \ln x, \quad x > 0.$$

Solution. Suppose $y = x^{-1}u$ is the solution of the nonhomogeneous equation. Then substituting

$$y' = x^{-1}u' - x^{-2}u \text{ and } y'' = x^{-1}u'' - 2x^{-2}u' + 2x^{-3}u,$$

in the differential equation, we get

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$$xu'' - u' = \ln x.$$

On substituting $w = u'$ above differential equation reduces to the linear equation

$$w' - \frac{1}{x}w = \frac{\ln x}{x},$$

1

which has integrating factor x^{-1}. Multiplying by integrating factor gives

$$\left(wx^{-1}\right)' = x^{-2}\ln x$$

1

and integrating, we get

$$wx^{-1} = -x^{-1}\ln x - x^{-1} + c_1.$$

It gives

1

$$u = -x\ln x + \frac{1}{2}c_1x^2 + c_2.$$

Hence, the general solution is

$$y = -\ln x + \frac{1}{2}c_1x + c_2x^{-1}.$$

5

Q2 b) (*) $\begin{cases} \dfrac{y''}{x^2-1} + (\tan x)\, y = e^x \\ y(0)=1, \ y'(0)=0 \end{cases}$

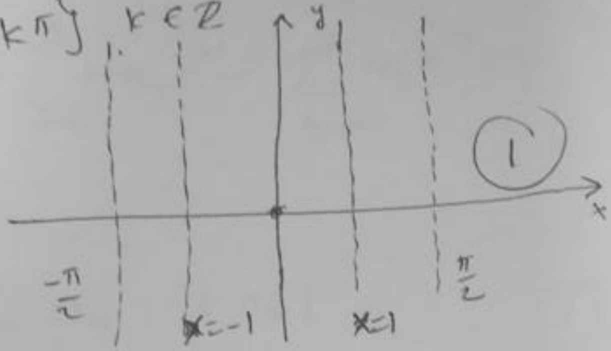
$a_2(x) = \dfrac{1}{x^2-1}$ is continuous on $\mathbb{R}-\{1,-1\}$

$a_1(x) = 0$ is continuous on $\mathbb{R}$ 2

$a_0(x) = \tan x$ is continuous on $\mathbb{R}-\{\frac{\pi}{2}+k\pi\}$ $k \in \mathbb{Z}$

$f(x) = e^x$ is cont on $\mathbb{R}$.

$a_2(x) \neq 0$ for $x \neq \pm 1$, and $a_2(x)$, $a_1(x)$, $a_0(x)$ and f are continuous $\mathbb{R}-\{\pm 1, \frac{\pi}{2}+k\pi\}$, $k \in \mathbb{Z}$



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Since $x_0 = 0 \in (-1,1)$ where all functions are continuous and $a_2(x) \neq 0$.

Hence (*) admits a unique sol on $I = (-1,1)$ 2

Q3 a) charact Eq: $m^3 - 3m + 2 = 0$

$(m-1)^2(m+2) = 0$ 1

$m_1 = m_2 = 1$, $m_3 = -2$

$y_p = x^2(Ax^2 + Bx + C)e^x + De^{-x} + \alpha \cos 2x + \beta \sin 2x$

2

=

Q_3 b) The functions $\frac{a_1(x)}{a_2(x)} = 3x$ and $\frac{a_0(x)}{a_2(x)} = -1$ are analytic for all $x \in \mathbb{R}$. Then the sol of the DE is of the form $y = \sum_{n=0}^{\infty} a_n x^n$ ($x_0 = 0$ ord point)

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

By substitution in the DE, we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3 \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 3 \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Thus $2a_2 - a_0 = 0$

$$a_{n+2} = \frac{1-3n}{(n+2)(n+1)} a_n \quad \forall n \geq 1$$ (1)

$a_2 = \frac{a_0}{2}$ or $y(0) = 2 \Rightarrow a_0 = 2$, hence $\boxed{a_2 = 1}$

$a_3 = \frac{-2}{6} a_1$, or $y'(0) = 0 \Rightarrow a_1 = 0$, hence $\boxed{a_3 = 0}$

$a_4 = \frac{-5}{12} a_2 = \frac{-5}{12}$, $a_5 = 0$, $a_6 = \frac{-11}{30} a_4 = \frac{11}{72}$

So $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots$

$$= 2 + x^2 - \frac{5}{12} x^4 + \frac{11}{72} x^6 + \cdots$$ (2)

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Consider the 2π-periodic, even function defined by $f(x) = 1 - \frac{2x}{\pi}$ if $x \in [0, \pi]$.

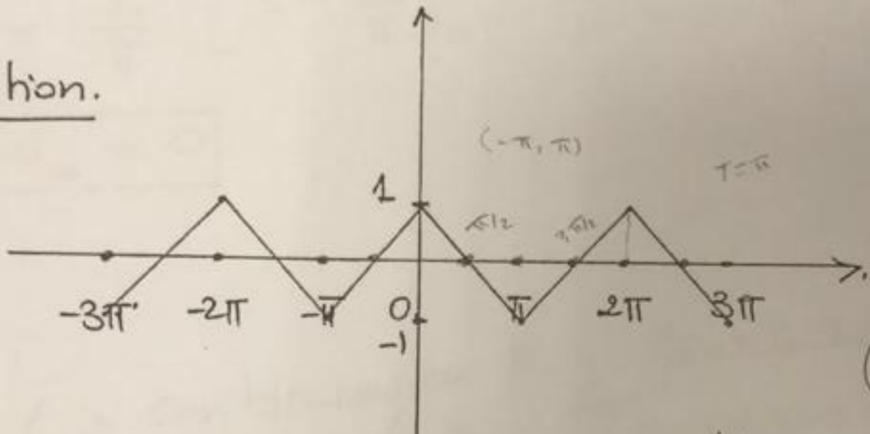
1) sketch the graph f on $(-3\pi, 3\pi)$.

2) Find the Fourier series of f

3) Deduce the Values of the following series: $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution.

1).



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2) The function f is even, then $b_n = 0$; $\forall n \geq 1$ and

- $a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt \qquad n \geq 1$

$$= \frac{2}{\pi} \int_0^{\pi} \underbrace{\left(1 - \frac{2}{\pi} t\right)}_{u} \underbrace{\cos nt}_{v'} \, dt$$

$$= \frac{2}{\pi} \left(\frac{1}{n} \left[\left(1 - \frac{2}{\pi} t\right) \sin nt \right]_0^{\pi} + \frac{2}{\pi n} \int_0^{\pi} \sin nt \, dt \right)$$

$$= \frac{4}{\pi^2 . n} \int_0^{\pi} \sin nt \, dt = \frac{-4}{\pi^2 n^2} \left[\cos nt \right]_0^{\pi}$$

$$= \frac{4}{\pi^2 n^2} \cdot \left(1-(-1)^n\right).$$

Hence;
$$\boxed{\begin{array}{l} \cdot\ a_{2n} = 0 \\ \cdot\ a_{2n+1} = \dfrac{8}{\pi^2 (2n+1)^2} \end{array}} \quad n = 0, 1 \ldots$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2}{\pi}t\right)dt$$
$$= \frac{2}{\pi}\left[t-\frac{t^2}{\pi}\right]_0^{\pi} = \frac{2}{\pi}\left(\pi - \frac{\pi^2}{\pi}\right) = 0$$
$$\boxed{a_0 = 0}$$

Since, f is continuous on $\mathbb{R}$; for all $x \in \mathbb{R}$
$$f(x) = FS(f, x) = \frac{a_0}{2} + \sum_{n \geq 1} a_n \cos nx + b_n \sin nx$$

$$f(x) = \frac{8}{\pi^2} \sum_{n \geq 0} \frac{\cos(2n+1)x}{(2n+1)^2} \quad (1)$$

3) For $x = 0$, $f(0) = 1$, then
$$1 = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \Rightarrow \boxed{\sum_{n \geq 0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{4} S + \frac{\pi^2}{8}. \quad (2)$$
$$\Rightarrow \frac{3}{4} S = \frac{\pi^2}{8} \Rightarrow \boxed{S = \frac{\pi^2}{6}}$$

Consider the function f defined by
$$f(x)=\begin{cases}1+x & \text{if } -1\le x\le 0\\ 1-x & \text{if } 0\le x\le 1\\ 0\ ; & \text{otherwise}\end{cases}$$

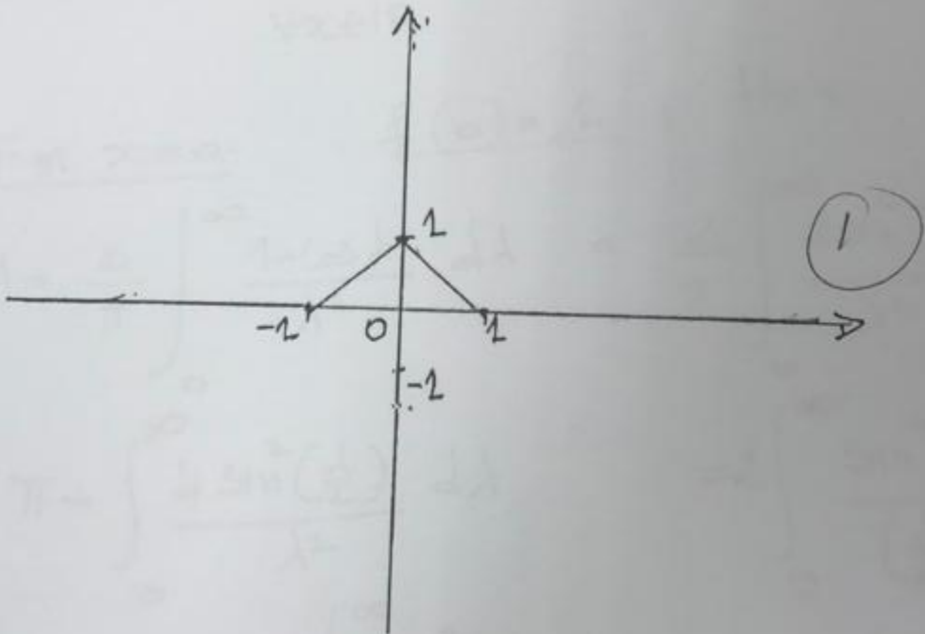
① Sketch the graph of f.

② Find the Fourier ~~series~~ integral of f.

③ Deduce the value of the improper convergent integral: $\displaystyle\int_0^{\infty}\frac{\sin^2\lambda}{\lambda^2}\,d\lambda$.

Solution.

1)



2) f is even, then $B(\lambda)=0$

and $A(\lambda)=2\displaystyle\int_0^1 f(t)\cos\lambda t\,dt$

$$=2\int_0^1\underbrace{(1-t)}_{u}\underbrace{\cos\lambda t}_{v'}\,dt=2\left(\left[\frac{(1-t)\sin(\lambda t)}{\lambda}\right]_0^1+\int_0^1\frac{\sin\lambda t}{\lambda}\,dt\right)$$

(4)

$$= \frac{2}{\lambda^2}\left[-\cos\lambda t\right]_0^1 = \boxed{\frac{2}{\lambda^2}\left(1-\cos\lambda\right)}$$

Then the Fourier integral of f is

$$FI(f,x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)\,d\lambda.$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(1-\cos\lambda)\cos(\lambda x)}{\lambda^2}\,d\lambda. \qquad 2$$

$= f(x)$ (Since f is continuous on $\mathbb{R}$).
$\forall x \in \mathbb{R}$

3 For $x=0$: $f(0)=1$; then.

$$1 = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^2}\,d\lambda = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\frac{\lambda}{2}}{\lambda^2}\,d\lambda \qquad 2$$

$$\Rightarrow \pi = \int_0^\infty \frac{4\sin^2(\frac{\lambda}{2})}{\lambda^2}\,d\lambda = 2\int_0^\infty \frac{\sin^2(\frac{\lambda}{2})}{(\frac{\lambda}{2})^2}\,d\frac{\lambda}{2}$$

$$= 2\int_0^\infty \frac{\sin^2\alpha}{\alpha^2}\,d\alpha. \qquad \left(\alpha = \frac{\lambda}{2}\right)$$

$$\Rightarrow \boxed{\int_0^\infty \frac{\sin^2\alpha}{\alpha^2}\,d\alpha = \frac{\pi}{2}} \ \#$$