

KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
TIME: 3H, FULL MARKS: 40, SI /21/04/1437 MATH 204

Question 1. [4,6] a) Determine the largest region for which the following IVP admits a unique solution

$$\sqrt{\frac{x}{y}}y' = \cos(x+y), \quad y \neq 0, \quad y(1) = 1$$

b) Solve the differential equations

$$[\cos x \ln(2y-8) + \frac{1}{x}]dx + \frac{\sin x}{y-4}dy = 0, \quad y > 4, \quad x \neq 0.$$

$$[x \cos\left(\frac{y}{x}\right) - y]dx + xdy = 0, \quad x > 0.$$

Question 2. a) [3,4]. Solve the initial value problem

$$(1-x)y' + xy = x(x-1)^2, \quad y(5) = 24.$$

b) Find the family of orthogonal trajectories for the family of curves

$$Cx^2 - y^2 = 1.$$

Question 3. a) [4,4]. Find the general solution of the differential equation

$$y'' - 2y' + y = \frac{e^x}{x}, \quad x > 0.$$

b) Write down the form of y_p for the solution of differential the equation

$$y^{(3)} + 4y' = 4 + xe^{-x} - e^x \sin x + 5 \cos 2x.$$

Question 4 [5]. Find the power series solution about the ordinary point $x_0 = 0$ for the differential equation $y'' - 2xy' + 2y = 0$.

Question 5. a) [5,5]. Let f be 2π -periodic function defined by:

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 \leq x < \pi \end{cases}$$

Sketch the graph of f on $[-2\pi, 2\pi]$, find the Fourier Series of f , and deduce the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

b) Consider the function

$$f(x) = \begin{cases} 0, & x < -1 \\ 1-x, & -1 \leq x < 1 \\ 0, & x \geq 1 \end{cases}$$

Sketch the graph of f , find its Fourier integral and deduce the value of $\int_0^{\infty} \frac{\sin \lambda}{\lambda} d\lambda$.

Question 1

(1) (a)

IVP:
$$\begin{cases} \sqrt{\frac{x}{y}} y' = \cos(x+y), & y \neq 0 \\ y(1) = 1 \end{cases}$$

$$y' = \frac{dy}{dx} = \cos(x+y) \left(\frac{x}{y}\right)^{-1/2} = f(x,y)$$

$$\frac{\partial f}{\partial y} = -\sin(x+y) \left(\frac{x}{y}\right)^{-1/2} - \frac{1}{2} \left(\frac{x}{y}\right)^{-3/2} \cdot \left(-\frac{x}{y^2}\right) \cos(x+y) \quad (1)$$

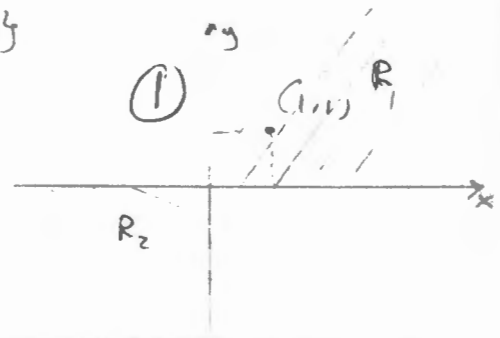
So it is clear that f and $\frac{\partial f}{\partial y}$ are continuous on the region

$$R = \left\{ (x,y) : \frac{x}{y} > 0 \right\} \text{ or}$$

$$R = \left\{ (x,y) : x > 0 \text{ and } y > 0 \right\} \cup \left\{ (x,y) : x < 0 \text{ and } y < 0 \right\}$$

But $(x,y) = (1,1) \in R_1 = \left\{ (x,y) : x > 0, \text{ and } y > 0 \right\}$

Then R_1 is the largest region s.t. the IVP admit a unique solution



(1) (b)

$$\int \left[\overset{M}{\cos x} \ln(2y-8) + \frac{1}{x} \right] dx + \frac{\overset{N}{\sin x}}{y-4} dy = 0, \quad x \neq 0, \quad y > 4$$

$$\frac{\partial M}{\partial y} = \cos x \frac{2}{2y-8} = \cos x \frac{1}{y-4}, \quad \frac{\partial N}{\partial x} = \frac{\cos x}{y-4} \quad (1)$$

Then the D.E is exact, hence \exists a function F of x and y s.t

$$\frac{\partial F}{\partial x} = \cos x \ln(y-4) + \cos x \ln 2 + \frac{1}{x}$$

$$\frac{\partial F}{\partial y} = \frac{\sin x}{y-4}$$

So
$$F(x,y) = \int \frac{\sin x}{y-4} dy = \sin x \ln(y-4) + f(x)$$

$$\frac{\partial F}{\partial x} = \cos x \ln(y-4) + f'(x) = \cos x \ln(y-4) + \cos x \cdot \ln 2 + \frac{1}{x}$$

$$f(x) = \sin x \ln 2 + \ln x + C$$

Thus the solution of the D.E. is

$$F(x,y) = \sin x \ln(y-4) + \sin(x) \cdot \ln 2 + \ln x + C = 0$$

2) $[x \cos(\frac{y}{x}) - y] dx + x dy = 0.$

This D.E. is homogeneous. We can put $u = \frac{y}{x} \Rightarrow y = xu$

$$dy = x du + u dx$$

$$[\cos(\frac{y}{x}) - \frac{y}{x}] dx + dy = 0$$

$$(\cos u - u) dx + x du + u dx = 0$$

$$(\cos u) dx + x du = 0 \Rightarrow \frac{dx}{x} + \frac{du}{\cos u} = 0; 0 < u = \frac{y}{x} < \frac{\pi}{2}$$

$$\ln x + \ln |\sec u + \tan u| = C$$

$$\ln x + \ln \left| \sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) \right| = C$$

Question 2

$$\begin{cases} (1-x)y' + xy = x(x-1)^2; x \neq 1; x > 1 \\ y(5) = 24 \end{cases}$$

The D.E. is linear. $y' + \frac{x}{1-x} y' = -x(1-x)$

$$\text{I.F. } \mu(x) = e^{\int \frac{x}{1-x} dx} = e^{\int (-1 + \frac{1}{1-x}) dx} = e^{-x - \ln|1-x|} = \frac{e^{-x}}{x-1}$$

$$y \mu(x) = y \frac{e^{-x}}{x-1} = \int \frac{x(1-x)e^{-x}}{x-1} dx = - \int x e^{-x} dx$$

$$y \frac{e^{-x}}{x-1} = - [-x e^{-x} - e^{-x}] + C = +e^{-x}(x+1) + C$$

$$y = +(x^2-1) + e^x(x-1)C$$

But $y(5) = 24 \Rightarrow 24 = 24 + e^5(4)C \Rightarrow C = 0$, then the solution of the IVP

is $y = x^2 - 1$

②⑥ $Cx^2 - y^2 = 1 \Rightarrow C = \frac{y^2+1}{x^2}$, we take the derivative implicitly
 two sides, $0 = \frac{2y\dot{y}x^2 - 2x(y^2+1)}{x^4} = 0$, hence

$$2y\dot{y}x^2 = 2x(y^2+1)$$

$$\dot{y}x = y^2+1 \text{ or } \dot{y} = \frac{y^2+1}{yx} = f(x,y) \quad (1)$$

Now we have to solve the D.E $\dot{y} = \frac{-1}{f(x,y)} = \frac{-(xy)}{y^2+1} \quad (1)$

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$$(y^2+1) dy + xy dx = 0$$

$$\frac{y^2+1}{y} dy + x dx = 0 \text{ or } \left(y + \frac{1}{y}\right) dy + x dx = 0$$

is orthogonal to $\frac{\frac{1}{2}(y^2+x^2) + \ln|y| = C}{Cx^2 - y^2 = 1}$, this family of curves

Question 3

③② $\ddot{y} - 2\dot{y} + y = \frac{1}{x}e^x; \quad x > 0$

1) $\ddot{y} - 2\dot{y} + y = 0, \quad y = e^{mx} \Rightarrow (m^2 - 2m + 1) = (m-1)^2 = 0, \quad m = 1, 1$

$$y_c = c_1 e^x + c_2 x e^x / \quad y_1 = e^x, \quad y_2 = x e^x \quad (1)$$

2) $y_p = u_1 y_1 + u_2 y_2 \quad s.t$

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$$\begin{cases} u_1'(e^x) + u_2'(x e^x) = 0 \\ u_1'(e^x) + u_2'(e^x + x e^x) = \frac{1}{x} e^x \end{cases} \Rightarrow \begin{cases} u_1' + x u_2' = 0 \\ u_1' + (1+x) u_2' = \frac{1}{x} \end{cases}$$

$$W = \begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix} = 1, \quad u_1' = \frac{\begin{vmatrix} 0 & x \\ \frac{1}{x} & 1+x \end{vmatrix}}{1} = -1 \Rightarrow \boxed{u_1 = -x} \quad (1)$$

$$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ 1 & \frac{1}{x} \end{vmatrix}}{1} = \frac{1}{x} \Rightarrow \boxed{u_2 = \ln x} \quad (1)$$

$$y_p = -x e^x + x e^x \ln x = \boxed{x e^x (\ln x - 1)} \quad (1)$$

Thus the solution of the D.E is

$$y = y_c + y_p = c_1 e^x + c_2 x e^x + x e^x (\ln x - 1)$$

(3) (b) $y + 4\ddot{y} = 4 + x e^{-x} - e^x \sin x + 5 \cos(2x)$

1) $\ddot{y} + 4y = 0 \Rightarrow m^3 + 4m = 0 \dots \textcircled{1}$

$m(m^2 + 4) = 0, m = 0, m = \pm 2i$ (1)

4) 0 is a root of $\textcircled{1} \Rightarrow 4 = 4e^{0m} \rightarrow Ax$

-1 is not a root of $\textcircled{1} \Rightarrow x e^{-x} \rightarrow (Bx + c) e^{-x}$ (1)

$1 + 2i$ is not a root of $\textcircled{1} \Rightarrow e^x \sin x \rightarrow D e^x \sin x + E e^x \cos x$

$0 + 2i$ is a root of $\textcircled{1} \Rightarrow -5 \cos(2x) \rightarrow Fx \cos(2x) + Gx \sin(2x)$ (2)

Then
$$y_p = Ax + (Bx + c) e^{-x} + D e^x \sin x + E e^x \cos x + Fx \cos(2x) + Gx \sin(2x)$$

is the general form of the D.E

Question (4) $y'' - 2xy' + 2y = 0$

$\frac{a_1}{a_2} = -2x, \frac{a_0}{a_2} = 2$ are two analytic functions on \mathbb{R}

the solution of the D.E is the form $y = \sum_{n=0}^{\infty} a_n x^n, x \in \mathbb{R}$

then

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0, x \in \mathbb{R}$$

$n-2=k \quad n=k \quad n=k$

(5)

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^{k+2} - \sum_{k=1}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} 2 a_k x^k = 0, x \in \mathbb{R}$$

$$(2a_2 + 2a_0) + \sum_{k=1}^{\infty} [(k+2)(k+2) a_{k+2} + 2a_k (-k+1)] x^k = 0$$

Hence $a_2 = -a_0$

$$a_{k+2} = \frac{2(k-1) a_k}{(k+2)(k+1)}, k \geq 1$$

(2)

$$k=1, a_3 = 0$$

$$k=2, a_4 = \frac{2}{4 \cdot 3} a_2 = \frac{-a_0}{6}$$

$$k=3, a_5 = \frac{2 \cdot 2}{5 \cdot 4} a_3 = 0 = a_3$$

$$k=4, a_6 = \frac{2 \cdot 3}{6 \cdot 5} a_4 = \frac{-a_0}{30}, \text{ and so on.}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 + a_1 x - a_2 x^2 + 0 - \frac{a_0}{6} x^4 + 0 - \frac{a_0}{30} x^6 + \dots$$

$$y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{6} x^4 - \frac{1}{30} x^6 - \dots \right) = a_1 y + a_0 y(x)$$

where

$$\begin{cases} u_1 = x \\ u_2 = 1 - x^2 - \frac{1}{6} x^4 - \frac{1}{30} x^6 - \dots \end{cases}$$

$$y = 1 - x^2 - \frac{1}{6} x^4 - \frac{1}{30} x^6 - \dots$$

$x \in \mathbb{R}$

Question 5

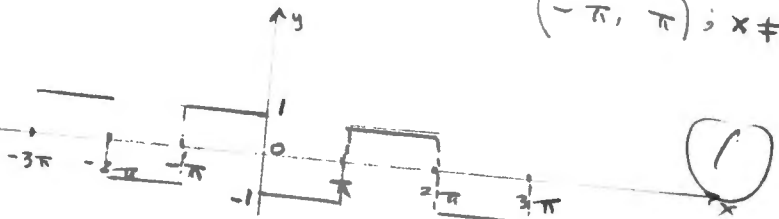
5, a

$$f(x) = \begin{cases} 1 & -\pi < x < 0 \\ -1 & 0 \leq x < \pi \end{cases}$$

f is an odd function on $(-\pi, \pi)$; $x \neq 0$

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Then $a_n = 0, n=0, 1, \dots$



$$b_n = \frac{-2}{\pi} \int_0^{\pi} (-1) \sin(nx) dx = \frac{2}{\pi} \left[\cos(nx) \right]_0^{\pi} = \frac{2}{\pi n} ((-1)^n - 1), \text{ hence}$$

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{n=1}^{\infty} \frac{2}{\pi n} ((-1)^n - 1) \sin(nx)$$

$-\pi < x < \pi$

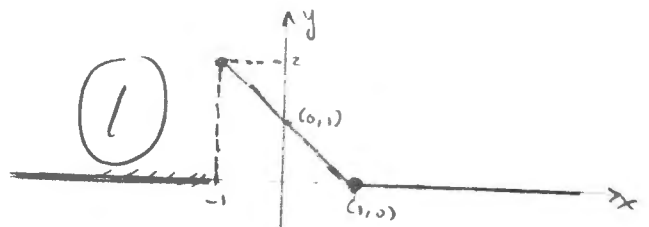
At $x = \frac{\pi}{2} \Rightarrow f\left(\frac{\pi}{2}^+\right) = -1 = \sum_{n=1}^{\infty} \frac{2}{\pi n} ((-1)^n - 1) \sin\left(n \frac{\pi}{2}\right)$

$$-1 = \sum_{n=1}^{\infty} \frac{-4}{\pi(2n-1)} \sin\left(\frac{2n-1}{2} \pi\right) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

So

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

$$\boxed{5b)} \quad f(x) = \begin{cases} 0 & ; x < -1 \\ 1-x & ; -1 \leq x < 1 \\ 0 & ; x \geq 1 \end{cases}$$



$$1) \quad A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx = \int_{-1}^1 (1-x) \cos \alpha x \, dx$$

$$= \left[(1-x) \frac{\sin \alpha x}{\alpha} \right]_{-1}^1 + \int_{-1}^1 \frac{\sin \alpha x}{\alpha} \, dx \quad \alpha > 0$$

$$= 0 - \frac{2 \sin(-\alpha)}{\alpha} - \left[\frac{\cos \alpha x}{\alpha^2} \right]_{-1}^1$$

$$= \frac{2}{\alpha} \sin \alpha - \frac{(\cos \alpha - \cos(-\alpha))}{\alpha^2} = \boxed{\frac{2}{\alpha} \sin \alpha}$$

$$2) \quad B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin(\alpha x) \, dx = \int_{-1}^1 (1-x) \sin \alpha x \, dx$$

$$= \left[(1-x) \left(-\frac{\cos \alpha x}{\alpha} \right) \right]_{-1}^1 - \int_{-1}^1 \cos \alpha x \, dx$$

$$= 0 + 2 \frac{\cos(-\alpha)}{\alpha} - \left[\frac{\sin \alpha x}{\alpha^2} \right]_{-1}^1$$

$$= \frac{2 \cos \alpha}{\alpha} - \left(\frac{\sin \alpha}{\alpha^2} - \frac{\sin(-\alpha)}{\alpha^2} \right)$$

$$\boxed{B(\alpha) = \frac{2 \cos \alpha}{\alpha} - \frac{2 \sin \alpha}{\alpha^2}}$$

$$\left[\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\pi} \int_0^{\infty} \left[\frac{2}{x} \sin x \cos \alpha x + \left(\frac{2 \cos \alpha}{\alpha} - \frac{2 \sin \alpha}{\alpha^2} \right) \sin \alpha x \right] d\alpha, \quad x \in \mathbb{R} \right]$$

At $x=0$, we have

$$\frac{f(0^+) + f(0^-)}{2} = \frac{1+1}{2} = 1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha}{\alpha} \, d\alpha \quad \text{or}$$

$$\boxed{\frac{\pi}{2} = \int_0^{\infty} \frac{\sin \alpha}{\alpha} \, d\alpha = \int_0^{\infty} \frac{\sin(\lambda)}{\lambda} \, d\lambda}$$