King Saud University, Mathematics Department
Math 204. Time: 3H, Full Marks: 40, 20/03/1437
Final Exam

Question 1. [4,4] **a)** Solve the initial value problem

$$\begin{cases} y' - xy\sin x = \frac{x\sin x}{y}, & y > 0 \\ y(0) = 1 \end{cases}$$

b) Find the general solution of the differential equation

$$xy' - y = x^3 - x, \quad x > 0$$

Question 2. [4,5] **a)** Show that $\mu(x,y) = x^{-5}y^{-2}$ is an integrating factor for the following equation and then solve it

$$(x^3y - 2y^2)dx + x^4dy = 0, \quad x \neq 0,\ y \neq 0.$$

b) The population of a town grows at a rate proportional to the population at any time. its initial population of 500 increases 15% in 10 years. Whatwill be the population in 40 years.

Question 3. [4,4] **a)** Solve the initial value problem

$$y'' + y = x + sinx, \quad y(0) = 1,\ y'(0) = -1.$$

b) Find the genral solution of the differential equation

$$xy'' - 3y' + \frac{5}{x}y = 0, \quad x > 0.$$

Question 4. [5] Find the power series solution near the ordinary point $x_0 = 0$ of the differential equation

$$y'' - (x+1)y' - y = 0.$$

Question 5. [5,5] **a)** Obtain the Fourier series of the function: $f(x) = \pi^2 - x^2$ on $(-\pi, \pi)$ and deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

b) Find the Fourier integral of the function

$$g(x) = \begin{cases} 0, & -\infty < x < -1 \\ -2, & -1 \leq x < 0 \\ 3, & 0 \leq x < 1 \\ 0, & x \geq 1, \end{cases}$$

and deduce that $\int_0^{\infty} \frac{\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}$

Complete solutions of final Exam

M.204 20/3/1437. First semester

Question 1. (a)
$$\begin{cases} y' - xy\sin x = \dfrac{x\sin x}{y}, & y>0 \\ y(0)=1 \end{cases}$$

First method: $y' = x\sin x\,(y+\frac{1}{y}) = x\sin x\,(\frac{y^2+1}{y})$

$$\frac{dy}{\frac{y^2+1}{y}} = x\sin x\,dx$$

$$\int \frac{y\,dy}{y^2+1} = \int x\sin x\,dx = [-x\cos x] + \int \cos x\,dx$$ (2)

$$\boxed{\tfrac{1}{2}\ln(y^2+1) + x\cos x - \sin x = c}$$

For $y(0)=1$, we have: $\frac{1}{2}\ln 2 + 0 - 0 = c$ (2)

Then the solution of the IVP is $\boxed{\frac{1}{2}\ln(y^2+1) + x\cos x - \sin x = \frac{1}{2}\ln 2}$

or $\ln(y^2+1) + 2x\cos x - 2\sin x = \ln 2$

Second method: $y' - x\sin x\, y = x\sin x\, y^{-1}$ is B.E.

$y'y - x\sin x\, y^2 = x\sin x$, we put $y^2 = u$, $2y'y = u'$

$yy' = \frac{u'}{2} \Rightarrow \frac{u'}{2} - x\sin x\, u = x\sin x$ or $u' - 2x\sin x\, u = 2x\sin x$

$$\mu(x) = e^{-\int 2x\sin x\,dx}$$
$$= e^{-2[-x\cos x + \sin x]}$$

$$u e^{-2(-x\cos x + \sin x)} = 2\int x\sin x\, e^{-2(-x\cos x + \sin x)}$$

$$u e^{-2(-x\cos x + \sin x)} = -e^{-2(-x\cos x + \sin x)} + c$$

$$y^2 = -1 + c e^{2(-x\cos x + \sin x)}$$

$y(0)=1 \Rightarrow 1 = -1 + c \Rightarrow c = 2$

$$y^2 + 1 = 2e^{2(-x\cos x + \sin x)}$$

$$\ln(y^2+1) = \ln 2 + 2(-x\cos x + \sin x)$$

$$\boxed{\tfrac{1}{2}\ln(y^2+1) + x\cos x - \sin x = \tfrac{1}{2}\ln 2}$$

(b) $xy' - y = x^3 - x$; $x > 0$

$y' - \frac{1}{x}y = x^2 - 1$ $\quad \mu(x) = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$ $\quad$ (1)

$y\frac{1}{x} = \int \frac{x^2-1}{x}\,dx = \int (x - \frac{1}{x})\,dx$ $\quad$ (3)

$y\frac{1}{x} = \frac{x^2}{2} - \ln x + c$

$$\boxed{y = \frac{x^3}{2} - x\ln x + cx}$$

Question (2) $\mu((x^3y - 2y^2)dx + x^4 dy = 0)$; $x \neq 0$, $y \neq 0$, $\mu(x,y) = \frac{1}{x^5y^2}$

(a)

$\left(\frac{1}{x^2y} - \frac{2}{x^5}\right)dx + \frac{1}{xy^2}dy = 0$ $\quad$ (1)

$\quad M \qquad\qquad N$

$\frac{\partial M}{\partial y} = \frac{-1}{x^2y^2}$, $\frac{\partial N}{\partial x} = \frac{-1}{x^2y^2}$ $\Rightarrow$ the D.E is exact $\Rightarrow$ (1)

$\exists\, F(x,y)$ s.t

$\frac{\partial F}{\partial x} = \frac{1}{x^2y} - 2x^{-5}$, $\quad \frac{\partial F}{\partial y} = \frac{1}{xy^2}$ $\quad$ (1)

$F(x,y) = \int \left(\frac{1}{x^2y} - 2x^{-5}\right)dx = \frac{-1}{xy} + \frac{1}{2}x^{-4} + \phi(y)$

$\frac{\partial F}{\partial y} = \frac{1}{xy^2} + \phi'(y) = \frac{1}{xy^2} \Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = c$

Then the sol of the D.E is $\quad$ (2)

$$\boxed{\frac{-1}{xy} + \frac{1}{2x^4} + C = 0}$$

(b) $\frac{dP}{P} = k\,dt \Rightarrow \int \frac{dP}{P} = \int k\,dt \Rightarrow \ln|P| = kt + c$ $\quad$ (1)

$|P| = e^{kt}\cdot e^{c} \Rightarrow P(t) = (\mp e^c)e^{kt}$

$P(t) = c_1 e^{kt}$; $c_1 = \pm e^c \neq 0$

$P(0) = 500$, $\quad P(t) = 500e^{kt}$ $\quad$ (1)

$P(10) = 500 + (500)\frac{15}{100} = 500 + 75 = 575$

$P(10) = 575 = 500e^{10k} \Rightarrow k = \frac{1}{10}\ln\left(\frac{575}{500}\right) = \frac{1}{10}\ln(1.15)$

$$\boxed{P(t) = 500e^{\frac{1}{10}\ln(1.15)t}}$$ (2)

$P(40) = 500e^{4\ln(1.15)} = 500(1.15)^4 \approx 875$ persons $\quad$ (1)

Question 3

(a) $\begin{cases} y'' + y = x + \sin x \\ y(0) = 1, \quad y'(0) = -1 \end{cases}$

1) $y'' + y = 0 \Rightarrow m^2 + 1 = 0, \quad m = \mp i$

$$y_c = c_1 \cos x + c_2 \sin x$$

2) $y_p = Ax + B + Cx\cos x + Dx\sin x$

$y_p' = A + C\cos x - Cx\sin x + D\sin x + Dx\cos x$

$y_p'' = -2C\sin x - Cx\cos x + 2D\cos x - Dx\sin x$ 1

$y_p'' + y_p = -2C\sin x - \cancel{Cx\cos x} + 2D\cos x - \cancel{Dx\sin x} + Ax + B + \cancel{Cx\cos x} + \cancel{Dx\sin x}$

$= x + \sin x$

$\Rightarrow A = 1, \quad B = 0, \quad D = 0, \quad -2C = 1 \Rightarrow C = -\frac{1}{2}$

2

Then $y_p = x - \frac{1}{2}x\cos x$

So the G. sol. of the D.E is $\boxed{y = c_1\cos x + c_2\sin x + x - \frac{1}{2}x\cos x}$

$y(0) = 1 \Rightarrow c_1 = 1$

$y'(x) = -c_1\sin x + c_2\cos x + 1 - \frac{1}{2}\cos x + \frac{1}{2}x\sin x$

$y'(0) = -1 \Rightarrow c_2 + 1 - \frac{1}{2} = -1 \Rightarrow c_2 = -\frac{3}{2}$ 1

Then the solution of the IVP is $\boxed{y = \cos x - \frac{3}{2}\sin x + x - \frac{1}{2}x\cos x}$

(b) $xy'' - 3y' + \frac{5}{x}y = 0 \quad x > 0$

$x^2y'' - 3xy' + 5y = 0, \quad y = x^m$

$m(m-1) - 3m + 5 = 0 \Rightarrow m^2 - 4m + 5 = (m-2)^2 + 1 = 0$

$(m-2)^2 = -1, \quad (m-2) = i$

$\boxed{m = 2 \mp i}$ 2

Then the sol of the D.Eq. is $\boxed{y = c_1x^2\cos(\ln x) + c_2x^2\sin(\ln x)}$ 2

[illegible]

$$y(x)=\sum_{0}^{\infty} a_n x^n ; \quad x \in \mathbb{R} \quad , \quad y'' - (x+1)y' - y = 0$$

$$y'' - (x+1)y' - y = \sum_{2}^{\infty} n(n-1) a_n x^{n-2} - (x+1) \sum_{1}^{\infty} n a_n x^{n-1} - \sum_{0}^{\infty} a_n x^n = 0$$

$$\sum_{2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{1}^{\infty} n a_n x^n - \sum_{1}^{\infty} n a_n x^{n-1} - \sum_{0}^{\infty} a_n x^n = 0$$

$n-2=k$, $n=k+2$ | $n=k$ | $n-1=k$, $n=k+1$ | $n=k$

$$\sum_{0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{1}^{\infty} k a_k x^k - \sum_{0}^{\infty} (k+1) a_{k+1} x^k - \sum_{0}^{\infty} a_k x^k = 0$$

$$(2a_2 - a_1 - a_0) + \sum_{1}^{\infty} \left[(k+2)(k+1) a_{k+2} - k a_k - (k+1) a_{k+1} - a_k \right] x^k = 0$$

$$2a_2 = a_1 + a_0 \quad \therefore \quad a_2 = \frac{1}{2} a_1 + \frac{1}{2} a_0$$

$$(k+2)(k+1) a_{k+2} = (k+1) a_{k+1} + (k+1) a_k \quad , \quad k \geq 1$$

$$\boxed{a_{k+2} = \frac{a_k + a_{k+1}}{k+2} ; \quad k \geq 1}$$

3

$k=1 \Rightarrow \quad a_3 = \frac{1}{3} a_1 + \frac{1}{3} a_2 = \frac{1}{3} a_1 + \frac{1}{6} a_1 + \frac{1}{6} a_0$

$$a_3 = \frac{1}{6} a_0 + \frac{1}{2} a_1$$

$k=2$

$$a_4 = \frac{1}{4} a_2 + \frac{1}{4} a_3 = \frac{1}{8} a_1 + \frac{1}{8} a_0 + \frac{1}{24} a_0 + \frac{1}{8} a_1$$

$$a_4 = \frac{1}{4} a_1 + \frac{1}{6} a_0$$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

2

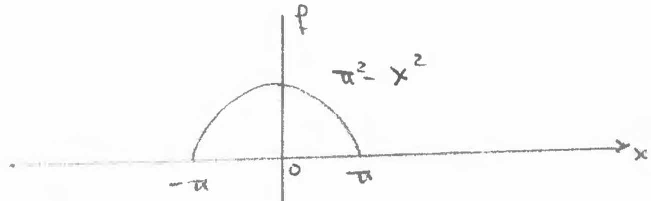
$= a_0 + a_1 x + \frac{1}{2} a_1 x^2 + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{2} a_1 x^3 + \frac{1}{4} a_1 x^4 + \frac{1}{6} a_0 x^4 \ldots$

$= a_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{6} x^4 \ldots \right] + a_1 \left(x + \frac{1}{2} x^2 + \ldots x^3 + \ldots x^4 \ldots \right)$

$= a_0 y_1 + a_1 y_2 \quad ; \quad x \in \mathbb{R}$

Question 5

(a)



The function $f(x) = \pi^2 - x^2$ is even on $(-\pi, \pi)$

then $b_n = 0$ 1

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\,dx = \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi}$$
$$= \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\pi^2 \cos nx\,dx - \frac{2}{\pi}\int_0^{\pi}x^2\cos(nx)\,dx$$

$$= \frac{2\pi}{n}\left[\sin nx\right]_0^{\pi} - \frac{2}{\pi}\left[x^2\frac{\sin nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}2x\frac{\sin nx}{n}\,dx$$

$$= \frac{4}{\pi n}\int_0^{\pi}x\sin nx\,dx = \frac{4}{\pi n}\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \frac{4}{\pi n^2}\int_0^{\pi}\cos(nx)\,dx$$

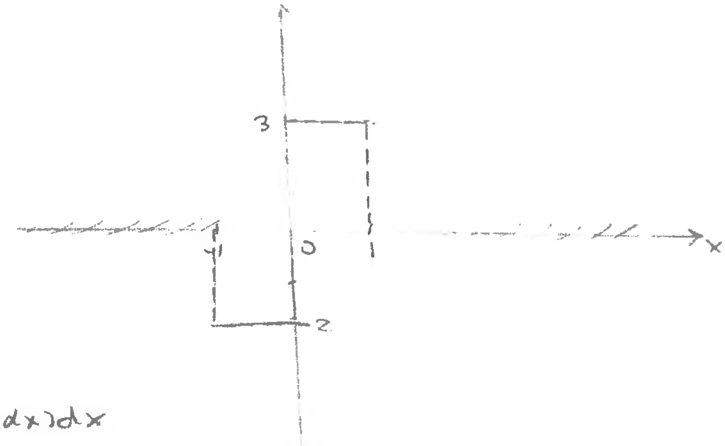
$$= \frac{4}{n^2}(-1)^{n+1} + \frac{4}{\pi n^3}\left[\sin nx\right]_0^{\pi} = \boxed{\frac{4(-1)^{n+1}}{n^2} = a_n}, \quad n = 1, 2, \ldots$$ 2

$$\boxed{\frac{f(x^+) + f(x^-)}{2} = f(x) = \pi^2 - x^2 = \frac{2}{3}\pi^2 + \sum_1^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos nx \qquad -\pi < x < \pi}$$ 2

At $x = 0$, we have:

$$f(0) = \pi^2 = \frac{2}{3}\pi^2 + \sum_1^{\infty}\frac{4(-1)^{n+1}}{n^2}$$
$$\frac{\pi^2}{3} = 4\sum_1^{\infty}\frac{(-1)^{n+1}}{n^2} \Rightarrow \boxed{\frac{\pi^2}{12} = \sum_1^{\infty}\frac{(-1)^{n+1}}{n^2}}$$

(b)



$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx$$

$$= -2\int_{-1}^{0}\cos\alpha x + 3\int_{0}^{1}\cos\alpha x\,dx = -2\left[\frac{\sin\alpha x}{\alpha}\right]_{-1}^{0} + 3\left[\frac{\sin\alpha x}{\alpha}\right]_{0}^{1}$$

$$= -2\left[0 - \frac{\sin(-\alpha)}{\alpha}\right] + 3\left(\frac{\sin\alpha}{\alpha}\right)$$ 1

$$= -2\frac{\sin\alpha}{\alpha} + 3\frac{\sin\alpha}{\alpha} =$$

$$\boxed{A(\alpha) = \frac{\sin\alpha}{\alpha}}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx = -2\int_{-1}^{0}\sin\alpha x\,dx + 3\int_{0}^{1}\sin\alpha x\,dx$$

$$= 2\left[\frac{\cos\alpha x}{\alpha}\right]_{-1}^{0} - 3\left[\frac{\cos\alpha x}{\alpha}\right]_{0}^{1}$$

$$= 2\frac{1-\cos(-\alpha)}{\alpha} - 3\frac{\cos\alpha - 1}{\alpha}$$ 0

$$\boxed{B(\alpha) = \frac{5-5\cos(\alpha)}{\alpha}}$$

$$\boxed{\frac{f(x^+)+f(x^-)}{2} = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin\alpha}{\alpha}\cos\alpha x + \frac{5-5\cos\alpha}{\alpha}\sin\alpha x\right]d\alpha}, \quad x\in\mathbb{R}$$

At $x=0$, we deduce

$$\frac{f(0^+)+f(0^-)}{2} = \frac{3-2}{2} = \frac{1}{2} = \frac{1}{\pi}\int_0^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha$$ or 2

$$\boxed{\frac{\pi}{2} = \int_0^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha}$$