King Saud University, Department of Mathematics
Math 204 (3H), 40/40, Final Exam17/3/36

Question 1[4,4] **a)** Determine and sketch the largest region of the xy-plane for which the following initial value problem has a unique solution

$$\begin{cases} (x-2)(x+3)y' = 4\ln y \\ y(-5) = 2. \end{cases}$$

b) Test if the following equation is exact, if it is not, find the appropriate integrating factor and solve it.

$$(3x^2+y)dx + (2x^2y - x)dy = 0.$$

Question 2[4,4,5] **a)** Solve the differential equation

$$\frac{dy}{dx} = \sqrt{3+x+y},$$

b) Solve the initial value problem

$$\begin{cases} xy\prime - 2(1+x+\sqrt{y})y = 0, \quad x>, y>0 \\ y(1) = 1. \end{cases}$$

c) A building loses heat in accordance with Newton's law of cooling. Assume the inside temperature is 70^0F when the heating system fails. After 2 hours the inside temperature drops to 40^0F. If the external temperature is 20^0F, compute the interior temperature after 4 hours.

Question 3[4,5] **a)** Use the variation of parameters method to solve the differential equation

$$y'' - 2y' + y = \frac{e^x}{x^2+1}.$$

b) Use power series method to solve the nonhomogeneous equation

$$y'' + xy' - 2y = x,$$

about the ordinary point $x = 0$.

Question 4[5,5] **a)** Let

$$f(x) = \begin{cases} 1+x, & -1 \le x \le 0 \\ -1+x, & 0 < x \le 1 \end{cases}$$

where $f(x+2) = f(x)\ \forall x \in \mathbb{R}$. Sketch the graph of $f(x)$ on $(-1,1)$ and find its Fourier series.

b) Find the Fourier integral of the function$f(x) = \begin{cases} -2, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \\ 0, & |x| > 1 \end{cases}$

and deduce that

$$\int_0^\infty \frac{(3-4\cos\lambda)\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}.$$

Answer Sheet for Final Exam

MATH 204/S1/2015

Q1 a) $\dfrac{dy}{dx} = \dfrac{4\ln y}{(x-2)(x+3)} = f(x,y)$

(i) f is continuous on $R_1 = \{(x,y)\in\mathbb{R}^2 : x\neq 2,\ x\neq -3,\ y>0\}$ 1

(ii) $\dfrac{\partial f}{\partial y} = \dfrac{4}{(x-2)(x+3)}\dfrac{1}{y}$ is continuous on 1

$R_2 = \{(x,y)\in\mathbb{R}^2 : x\neq 2,\ x\neq -3,\ y\neq 0\}$

So f and $\dfrac{\partial f}{\partial y}$ are continuous on

$R = R_1 \cap R_2 = R_1$

1

$(-5,2)\in R^* = \{(x,y)\in\mathbb{R}^2 : x<-3,\ y>0\}$ where f and $\dfrac{\partial f}{\partial y}$ are continuous, then R^* is the largest region for the I.V.P admits a unique solution. 1

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b) $M(x,y) = 3x^2+y$, $N(x,y) = 2x^2y - x$

$\dfrac{\partial M}{\partial y} = 1$, $\dfrac{\partial N}{\partial x} = 4xy-1 \Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ the DE 1 is not exact, but $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{2-4xy}{2x^2y-x} = -\dfrac{2}{x}$

$\Rightarrow \mu(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = \dfrac{1}{x^2}$ 1

Multiply the DE by $\mu(x) = x^{-2}$, we obtain

$$\left(3 + \frac{y}{x^2}\right) dx + \left(2y - \frac{1}{x}\right) dy = 0 \quad (\text{Exact Eq})$$

$$\Rightarrow \exists\ F(x,y) : \begin{cases} \frac{\partial F}{\partial x} = 3 + \frac{y}{x^2} \longrightarrow (1) \\ \frac{\partial F}{\partial y} = 2y - \frac{1}{x} \longrightarrow (2) \end{cases}$$

From (1), we have $F(x,y) = 3x - \frac{y}{x} + h(y)$ 1

$$\Rightarrow \frac{\partial F}{\partial y} = -\frac{1}{x} + h'(y) \longrightarrow (3)$$

From (2) and (3), we deduce that

$$h'(y) = 2y \Rightarrow h(y) = y^2 + c_1$$ 1

Thus $F(x,y) = 3x - \frac{y}{x} + y^2 = C$

Q2 a) $y' = \sqrt{3+x+y}$, let $u = 3+x+y \Rightarrow u' = 1 + y' \Rightarrow$ 1

$u' - 1 = \sqrt{u}$ or $\frac{1}{1+\sqrt{u}} du = dx \Rightarrow \int \frac{du}{1+\sqrt{u}} = x + c$ 1

Now let $1+\sqrt{u} = w \Rightarrow u = (w-1)^2 \Rightarrow du = 2(w-1)\,dw$,

hence $\int \frac{2(w-1)dw}{w} = x + c \Rightarrow 2w - 2\ln w = x + c$ 1

$$\Rightarrow 2\left(1+\sqrt{u}\right) - 2\ln\left(1+\sqrt{u}\right) = x + c$$

or $2\left(1 + \sqrt{1+x+y}\right) - 2\ln\left(1+\sqrt{1+x+y}\right) = x + c$ 1

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b) $xy' - 2(1+x+\sqrt{y})y = 0 \Rightarrow xy' - 2(1+x)y = 2y^{3/2}$

$\Rightarrow y' - 2\left(\frac{1}{x}+1\right)y = \frac{2}{x} y^{3/2}$ (B.E) 1

$$\Rightarrow y^{-3/2}y' - 2\left(\frac{1}{x}+1\right)y^{-\frac{1}{2}} = \frac{2}{x}$$

Let $w = y^{-\frac{1}{2}} \Rightarrow w' = -\frac{1}{2} y^{-\frac{3}{2}} y'$

or $-2w' - 2\left(\frac{1}{x}+1\right)w = \frac{2}{x}$ 1

$\Rightarrow w' + \left(\frac{1}{x}+1\right)w = -\frac{1}{x}$ (Linear Eq) $\longrightarrow (*)$

$\mu(x) = e^{\int\left(\frac{1}{x}+1\right)dx} = xe^{x}$

Multiply (*) by $\mu(x)$, we obtain

$\frac{d}{dx}\left(xe^{x}w\right) = \cancel{-}e^{x}$ 1

$\Rightarrow \frac{xe^{x}}{\sqrt{y}} = \cancel{-}e^{x} + C$

$y(1)=1 \Rightarrow e = C - e \Rightarrow C = 2e$ 1

Hence $\frac{xe^{x}}{\sqrt{y}} = -e^{+x} + 2e$

Q4 c) $T(0)=70$, $T(2)=40$, $T_s = 20$

$\frac{dT}{dt} = k(T-T_s) \Rightarrow T(t) = T_s + Ce^{kt}$ 1

$T(0)=70 \Rightarrow 70+20+C \Rightarrow C=50 \Rightarrow T(t) = 20 + 50e^{kt}$ 4

$T(2)=40 \Rightarrow 40 = 20 + 50e^{2k} \Rightarrow e^{2k} = \frac{2}{5}$

$T(4) = 20 + 50e^{4k} = 20 + 50\left(e^{2k}\right)^2 = 20 + 50\left(\frac{2}{5}\right)^2$

$= 20 + \frac{50\times 4}{25}$

$= 28$

Q_3 a) $y'' - 2y' + y = \dfrac{e^x}{x^2+1}$

$y_g = y_{gh} + y_p$

(HE) $y'' - 2y' + y = 0$, charact Eq $m^2 - 2m + 1 = 0 \Rightarrow m_1 = m_2 = 1$

$\Rightarrow \quad y_{gh} = C_1 e^x + C_2 x e^x$ 1

$y_p = C_1(x) e^x + C_2(x) x e^x$

$$\begin{cases} C_1'(x) e^x + C_2'(x) x e^x = 0 \\ C_1'(x) e^x + C_2'(x)(1+x) e^x = \dfrac{e^x}{x^2+1} \end{cases}$$

$$W = \begin{vmatrix} e^x & x e^x \\ e^x & (1+x) e^x \end{vmatrix} = e^{2x}$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & x e^x \\ \frac{e^x}{x^2+1} & (1+x) e^x \end{vmatrix}}{e^{2x}} = \frac{-x}{x^2+1} \Rightarrow C_1(x) = -\frac{1}{2} \ln(1+x^2)$$ 1

$$C_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x^2+1} \end{vmatrix}}{e^{2x}} = \frac{1}{x^2+1} \Rightarrow C_2(x) = \tan^{-1} x$$ 1

$$y_p = -\frac{1}{2} \ln(1+x^2) e^x + (\tan^{-1} x) x e^x$$ 1

$$y_g = (C_1 + C_2 x) e^x - \frac{1}{2} \ln(1+x^2) e^x + (x \tan^{-1} x) e^x.$$

b) Let $y = \sum_{n=0}^{\infty} a_n x^n$, Then we have

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = x$$ 1

with appropriate indexing, we have

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = x$$ 1

Separating the terms of x^0 and x^1 and equating, we get (5)

$$a_2 = a_0 , \qquad a_3 = \frac{1}{3!} + \frac{a_1}{3!}$$ and the

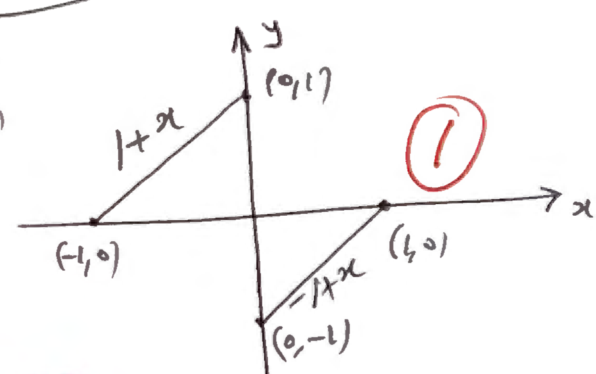
recurrence relation $$a_{n+2} = \frac{(2-n)}{(n+1)(n+2)} a_n , \quad n = 2, 3 \ldots$$ (1)

$n=2$, $a_4 = 0$, $n=3$, $a_5 = -\frac{1}{5!} - \frac{a_1}{5!}$, $n=4$, $a_6 = 0$

$n=5$, $a_7 = \frac{3}{7!} + \frac{3a_1}{7!}$, ...

Hence $$y = a_0(1+x^2) + a_1\left(x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots\right) + \left(\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{3x^7}{7!} + \cdots\right)$$ (2)

Q5 a) $$f(x) = \begin{cases} 1+x , & -1 \le x < 0 \\ -1+x , & 0 < x \le 1 \end{cases}$$



(1)

f is odd on $[-1,1]$

So $a_n = 0$, $n = 0, 1, 2, \ldots$

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$ (1)

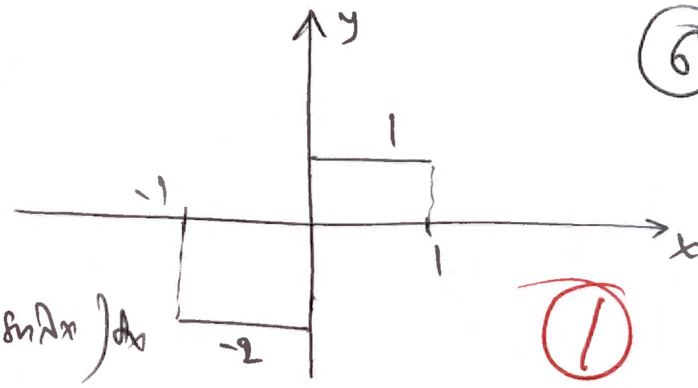
$$= 2\int_0^1 (-1+x)\sin n\pi x\,dx = 2\left[\frac{(1-x)\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{\cos(n\pi x)}{n\pi}dx\right]$$

$$= -\frac{2}{n\pi} + \frac{1}{n^2\pi^2}\sin n\pi x\Big|_0^1$$

$$= -\frac{2}{n\pi}$$ (2)

$$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n\pi}\sin(n\pi x), \quad x \in [-1,1]$$ (1)

Q5 b) $f(x) = \begin{cases} -2, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \\ 0 & |x| > 1 \end{cases}$ 6



$$f(x) = \frac{1}{\pi} \int_0^\infty (A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x)\, d\lambda$$ 1

$$A(\lambda) = \int_{-\infty}^{\infty} f(t) \cos(\lambda t)\, dt = \int_{-1}^{0} -2 \cos(\lambda t)\, dt + \int_0^1 \cos(\lambda t)\, dt$$ 1

$$= -\frac{\sin \lambda}{\lambda}.$$

$$B(\lambda) = \int_{-1}^{0} -2 \sin(\lambda t)\, dt + \int_0^1 \sin(\lambda t)\, dt$$

$$= +2 \left.\frac{\cos \lambda t}{\lambda}\right|_{-1}^{0} - \left.\frac{\cos \lambda t}{\lambda}\right|_0^1 = \frac{3 - 3\cos \lambda}{\lambda}$$ 1

Hence $$f(x) = \frac{1}{\pi} \int_0^\infty \left[-\frac{\sin \lambda}{\lambda} \cos(\lambda x) + \frac{3(1 - \cos \lambda)}{\lambda} \sin(\lambda x) \right] d\lambda$$

At $x = 1$: $$\frac{0+1}{2} = \frac{1}{2} = \frac{1}{\pi} \int_0^\infty \left[-\sin \lambda \cos \lambda + 3 \sin \lambda - 3 \cos \lambda \sin \lambda \right] d\lambda$$ 2

$$\Leftrightarrow \frac{\pi}{2} = \int_0^\infty \frac{\sin \lambda (3 - 4 \cos \lambda)}{\lambda}\, d\lambda$$

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