KING SAUD UNIVERSITY
DEPARTMENT OF MATHEMATICS
TIME: 3H, FULL MARKS: 40, SI /08/03/1435, MATH 204

Question 1. [5] Find and sketch the largest region in the $xy-$plane for which the following initial value problem admits a unique solution

$$(2x-1)(x+3)dy + \sqrt{y}dx = 0, \quad y(-5) = 5.$$

Question 2. a) [4]. A pot of liquid is put on the sotre to boil. The temperature of the liquid reaches $170^0 F$ and then the pot is taken of the burner and placed on a counter in the kitchen where the temperature is $76^0 F$. After 2 minutes the temperature of the liquid is $123^0 F$. How long before the temperature of the liquid in the pot will be $84^0 F$.

b) [4]. If $y_1 = \frac{e^x}{x}$ is a solution of the differential equation:

$$y'' + \frac{2}{x}y' - y = 0,$$

then find its general solution.

Question 3. a) [4]. By using the undetermined coefficients method, give only the form of the particular solution y_p of the differential equation

$$y^{(4)} - y = xe^x - x^2 \sin x + 7^x$$

b) [5]. Solve the system of linear differential equations

$$\begin{cases} \frac{dx}{dt} + \frac{dy}{dt} + 2y = 0 \\ \frac{dx}{dt} - 3x - 2y = 0. \end{cases}$$

Question 4. a) [4]. By using the power series method, find the solution of the differential equation : $y'' - 2xy' - y = 0$, about the ordinary point $x_0 = 0$.

b) [4] Solve the Cauchy Euler equation:

$$x^2y'' - 3xy' + 3y = 2x^4e^x, \qquad x > 0.$$

Question 5. a) [5]. Expand in Fourier cosine series, the following function

$$f(x) = x + \pi, \quad 0 < x < \pi.$$

Decuce that: $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

b) [5]. Find the Fourier integral representation for the function

$$f(x) = \begin{cases} 0, & x < 0 \\ \cos x, & 0 \le x \le \frac{\pi}{2} \\ 0, & x > \frac{\pi}{2} \end{cases}$$

Deduce that $\int_0^{\infty} \frac{\cos(\lambda\pi/2)}{1-\lambda^2} d\lambda = \frac{\pi}{2}, \quad \lambda \neq 1.$

$$\cos a \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}, \quad \sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b, \quad \sin(a+b) = \sin a \cos b + \sin b \cos a.$$

$\boxed{\text{Complete solution of final Exam} \quad \text{M. 204} \quad \text{F. Sem. 1434/1435}}$

$\boxed{Q1}$ (a)

$$\frac{dy}{dx} = f(x,y) = \frac{1}{(2x-1)(x+3)} \cdot \sqrt{y}$$

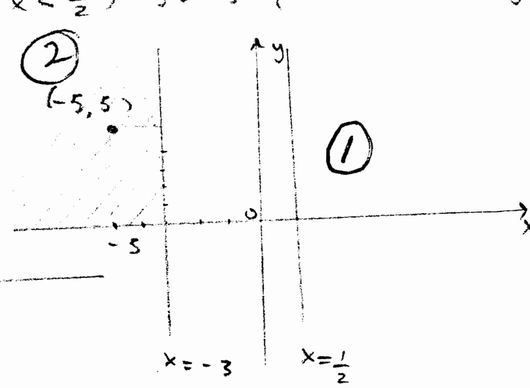
$$\frac{\partial f}{\partial y} = \frac{-1}{(2x-1)(x+3)} \cdot \frac{1}{2\sqrt{y}} \qquad (1)$$

f and $\frac{\partial f}{\partial y}$ are continuous on $R = \{(x,y);\ x \neq \frac{1}{2},\ x \neq -3\ \&\ y > 0\}$. (1)

But $R = \{(x,y);\ x < -3,\ y > 0\} \cup \{(x,y);\ -3 < x < \frac{1}{2},\ y > 0\} \cup \{(x,y);\ x > \frac{1}{2},\ y > 0\}$. (2)

As $(-5, 5) \in R_1 = \{(x,y);\ x < -3\ \&\ y > 0\}$,

then R_1 is the largest region in xy-plane for which the IVP has a unique solution (1)



Q_1 (b)

$$(x+2)^2 y' = 5 - 8y - 4xy; \qquad x > -2$$

$$(x+2)^2 y' + 4y(x+2) = 5$$

$$y' + \frac{4}{x+2} y = \frac{5}{(x+2)^2}$$

$$\mu(y) = e^{\int \frac{4}{x+2} dx} = e^{\ln (x+2)^4} = (x+2)^4$$

$$y\mu(x) = y(x+2)^4 = \int \frac{5}{(x+2)^2} (x+2)^4 dx = \int 5 (x+2)^2 dx$$

$\boxed{y(x+2)^4 = \frac{5}{3}(x+2)^3 + C}$ or $\boxed{y = \frac{5}{3} \frac{1}{(x+2)} + c\left(\frac{1}{(x+2)^4}\right)}$

is the General solution of the D.E

$\boxed{Q2}$ (a) We have $\frac{dT}{dt} = k(T - T_s) \Rightarrow \frac{dT}{T - T_s} = k\,dt$

$$\ln |T - T_s| = kt \Rightarrow \boxed{T(t) = T_s + c\,e^{kt}} \qquad (1)$$

But $T_s = 76$, $T(0) = 170$, $T(2) = 123$, then

$$T(t) = 76 + Ce^{kt}$$

$$T(0) = 76 + C = 170 \Rightarrow \boxed{C = 94} \qquad (1)$$

$$T(t) = 76 + 94e^{kt}$$

$$T(2) = 123 = 76 + 94e^{2k} \Rightarrow 47 = 94e^{2k}$$

hence $\frac{1}{2} = e^{2k}$ or $k = \frac{1}{2}\ln(\frac{1}{2}) \simeq -0.3465$ (1)

So $\boxed{T(t) = 76 + 94e^{-0.3465t}}$

For, $T(t) = 84 = 76 + 94e^{-0.3465t} \Rightarrow \frac{8}{94} = e^{-0.3465t}$

then $t = \frac{\ln(\frac{4}{47})}{-0.3465} \simeq \boxed{7.11}$ minutes (1)

Q2, (b) $y'' + \frac{2}{x}y' - y = 0$, $y_1 = \frac{e^x}{x}$ is a given solution, we can use

the formula $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx$, $e^{-\int p(x)dx} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$ (2)

$$= \frac{e^x}{x} \int \frac{1/x^2}{e^{2x}/x^2} dx = \frac{e^x}{x} \int e^{-2x} dx$$

$$y_2 = \frac{-1}{2} \frac{e^x}{x} e^{-2x} = \boxed{\frac{-1}{2x} e^{-x}} \text{ or } \boxed{y_2 = \frac{1}{x} e^{-x}}$$

Then the G. solution of the D.E is (2)

$$\boxed{y = c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{1}{x} e^{-x}\right)}$$

We can find the solution by using the reduction of order: let $y = y_1 u$

or $y = \frac{e^x}{x} u \Rightarrow y' = \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)u + \frac{e^x}{x}u'$

$$\boxed{y' = \frac{e^x}{x}u - \frac{e^x}{x^2}u + \frac{e^x}{x}u'}$$

$$y'' = \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)u + \frac{e^x}{x}u' - \frac{e^x}{x^2}u + \frac{2}{x^3}e^x u - \frac{e^x}{x^2}u'$$
$$+ \frac{e^x}{x}u' - \frac{e^x}{x^2}u' + \frac{e^x}{x}u''$$

$$\boxed{y'' = \left(\frac{e^x}{x} - 2\frac{e^x}{x^2} + \frac{2}{x^3}e^x\right)u + \left(-2\frac{e^x}{x^2} + 2\frac{e^x}{x}\right)u' + \frac{e^x}{x}u''}$$

$$y'' + \frac{2}{x}y' - y = \left(\frac{e^x}{x} - 2\frac{e^x}{x^2} + \frac{2}{x^3}e^x\right)u + \left(-2\frac{e^x}{x^2} + 2\frac{e^x}{x}\right)u' + \frac{e^x}{x}u'' +$$
$$\left(2\frac{e^x}{x^2} - \frac{2e^x}{x^3}\right)u + \frac{2e^x}{x^2}u' - \frac{e^x}{x}u = 0$$

$\left(2\frac{e^x}{x}\right)u' + \frac{e^x}{x}u'' = 0 \Rightarrow u'' + 2u' = 0$, let $u' = w \Rightarrow$

$w' + 2w = 0 \Rightarrow \frac{dw}{w} = -2dx \Rightarrow w = c_1 e^{-2x}$

"$u' = w = c_1 e^{-2x} \Rightarrow u = -\frac{c_1}{2} e^{-2x} + c_2$

$y = \frac{e^x}{x} u = \boxed{-\frac{c_1}{2} \frac{e^{-x}}{x} + c_2 \left(\frac{e^x}{x}\right) = y}$" or

$\boxed{y = c_3 \left(\frac{e^{-x}}{x}\right) + c_2 \left(\frac{e^x}{x}\right)}$ is the G. sol. of the D.E.

Q3 (a) $y^{(4)} - y = xe^x - x^2\sin x + 7^x = xe^x - x^2\sin x + e^{x\ln 7}$

For $y^{(4)} - y = 0 \Rightarrow m^4 - 1 = (m^2+1)(m^2-1) = (m^2+1)(m-1)(m+1) = 0$

$m = \mp i,\quad m = 1,\quad m = -1$ (2)

$$\boxed{y_p = x(Ax+B)e^x + (a_1x^2+a_2x+a_3)x\sin x + (b_1x^2+b_2x+b_3)x\cos x + Ce^{x\ln 7}}$$ (2)

Q3 (b)

$$\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} + 2y = 0 \\ \dfrac{dx}{dt} - 3x - 2y = 0 \end{cases} \Rightarrow \begin{cases} Dx + (D+2)y = 0 & (1) \\ (D-3)x - 2y = 0 & (2) \end{cases}$$

We have to eliminate x: $(D-3)\left[Dx + (D+2)y\right] = 0$

$+ \quad -D\left[(D-3)x - 2y\right] = 0$ (1)

$(D-3)(D+2)y + 2Dy = 0$ or

$x'' + x'$

$y'' + y' - 6y = 0$, $y = e^{mt}$

$m^2 + m - 6 = (m+3)(m-2) = 0 \Rightarrow m = -3,\ m = 2$

$\boxed{y(t) = c_1e^{2t} + c_2e^{-3t}}$. (3) $\quad 1\frac{1}{2}$

But $x'(t) = -y' - 2y = -(2c_1e^{2t} - 3c_2e^{-3t}) - 2c_1e^{2t} - 2c_2e^{-3t}$

$x'(t) = -4c_1e^{2t} + c_2e^{-3t} \Rightarrow$

$\boxed{x(t) = -2c_1e^{2t} - \frac{1}{3}c_2e^{-3t} + c_4}$ (4) $\quad 1\frac{1}{2}$

Now we replace (3) and (4) in (2), we obtain

$x'(t) - 3x - 2y = -4c_1e^{2t} + c_2e^{-3t} + 6c_1e^{2t} + c_2e^{-3t} - 3c_4 - 2c_1e^{2t} - 2c_2e^{-3t} = 0$

$\Rightarrow c_4 = 0$

So the solution of the system is $\begin{cases} x(t) = -2c_1e^{2t} - \frac{1}{3}c_2e^{-3t} \\ y(t) = c_1e^{2t} + c_2e^{-3t} \end{cases}$ (1)

Remark: we can find the solution for this system by eliminating y.

Q4

(a) $y'' - 2xy' - y = 0, \quad y = \sum_{n=0}^{\infty} a_n x^n, \quad x \in \mathbb{R}$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{0}^{\infty} a_n x^n = 0$$

$n = k+2 \qquad n = k \qquad n = k$

$$\sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2} x^k - \sum_{k=1}^{\infty} 2k a_k x^k - \sum_{0}^{\infty} a_k x^k = 0$$ 1

$$(2a_2 - a_0) + \sum_{k=1}^{\infty} [(k+1)(k+2)a_{k+2} - 2k a_k - a_k] x^k = 0$$

$\Rightarrow \boxed{a_2 = \frac{1}{2}a_0}, \quad \boxed{a_{k+2} = \frac{(1+2k)a_k}{(k+1)(k+2)}}; \quad k \geq 1$ 1

$k=1, \quad a_3 = \frac{3a_1}{2.3} = \boxed{\frac{1}{2}a_1}$

$k=2, \quad a_4 = \frac{5}{3.4}a_2 = \frac{5}{3.4} \cdot \frac{1}{2}a_0 = \boxed{\frac{5}{24}a_0}$

$k=3, \quad a_5 = \frac{7}{4.5}a_3 = \frac{7}{4.5} \cdot \frac{1}{2}a_1 = \boxed{\frac{7}{40}a_1}$ and so on...

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$y = a_0 + a_1 x + \frac{1}{2}a_0 x^2 + \frac{1}{2}x^3 a_1 + \frac{5}{24}x^4 a_0 + \frac{7}{40}x^5 a_1 + \cdots$ 2

$y = a_0 \underbrace{[1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots]}_{y_1(x)} + a_1 \underbrace{[x + \frac{1}{2}x^3 + \frac{7}{40}x^5 + \cdots]}_{y_2(x)}; \quad x \in \mathbb{R}$

$y = a_0 y_1(x) + a_1 y_2(x)$

Q4 (b) $x^2 y'' - 3xy' + 3y = 2x^4 e^x; \quad x > 0$

1) $x^2 y'' - 3xy' + 3y = 0, \quad y = x^m$

$m(m-1) - 3m + 3 = 0 \Rightarrow m^2 - 4m + 3 = 0, \quad (m-1)(m-3) = 0$ 1

$m = 1, \ m = 3 \Rightarrow y_c = c_1 x + c_2 x^3; \quad y_1 = x, \quad y_2 = x^3$

2) $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2 e^x$

$y_p = u_1 y_1 + u_2 y_2,$

$$\begin{cases} u_1'(x) + u_2'(x^3) = 0 \\ u_1'(1) + u_2'(3x^2) = 2x^2e^x \end{cases}$$

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x, \quad u_1' = \frac{W_1}{W} = \frac{-2x^5e^x}{2x^3} = -x^2e^x$$

$$u_1 = -\int x^2e^x\,dx = \boxed{-x^2e^x + 2xe^x - 2e^x}$$

$$u_2' = \frac{W_2}{W} = \frac{2x^3e^x}{2x^3} = e^x \Rightarrow \boxed{u_2 = e^x}$$

$$y_p = (-x^2e^x + 2xe^x - 2e^x)x + e^x(x^3)$$

$$\boxed{y_p = 2x^2e^x - 2xe^x = 2xe^x(x-1)}$$

$$\boxed{y = y_c + y_p = c_1x + c_2x^3 + 2xe^x(x-1)}$$

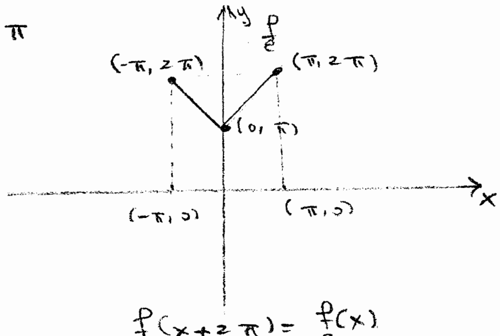
$\boxed{Q4}$ ⓐ

$T = \pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(x+\pi)\,dx$$

$$a_0 = \frac{2}{\pi}\left[\pi x + \frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\cdot\left(\frac{3}{2}\pi^2\right)$$

$$\boxed{a_0 = 3\pi}$$



$f_e(x+2\pi) = f_e(x)$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(x+\pi)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[(x+\pi)\frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin nx}{n}\,dx$$

$$= \frac{2}{\pi n^2}\left[\cos nx\right]_0^{\pi} = \boxed{\frac{2}{\pi n^2}\left[(-1)^n - 1\right]}$$

$$\boxed{f(x) = x + \pi = \frac{3\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos(nx);} \quad 0 < x < \pi$$

At $x = 0 \Rightarrow$

$$\pi = \frac{3\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left((-1)^n - 1\right) = \frac{3\pi}{2} + \sum_{n=1}^{\infty}\frac{2(-2)}{\pi(2n-1)^2}$$

$$\Rightarrow -\frac{\pi}{2} = -\frac{4}{\pi}\sum_{1}^{\infty}\frac{1}{(2n-1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{1}^{\infty}\frac{1}{(2n-1)^2}$$

or $$\boxed{\frac{\pi^2}{8} = \sum_{0}^{\infty}\frac{1}{(2n+1)^2}}$$

$\boxed{Q4_{(b)}}$ $f(x) = \begin{cases} 0; & x<0 \\ \cos x & 0 \le x \le \frac{\pi}{2}, \\ 0; & x > \frac{\pi}{2} \end{cases}$

$\boxed{\alpha = \lambda}$

$$A(\alpha) = \int_0^{\frac{\pi}{2}} \cos x \cos(\alpha x)\, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} [\cos(\alpha+1)x + \cos(1-\alpha)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha+1)x}{1+\alpha} + \frac{\sin(1-\alpha)x}{1-\alpha}\right]_0^{\frac{\pi}{2}}; \quad \alpha \neq 1$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha+1)\pi/2}{1+\alpha} + \frac{\sin(1-\alpha)\pi/2}{1-\alpha}\right]$$ 1

$$= \frac{1}{2}\left[\frac{\cos(\alpha\frac{\pi}{2})}{1+\alpha} + \frac{\cos(\alpha\pi/2)}{1-\alpha}\right] = \frac{\cos(\alpha\pi/2)}{1-\alpha^2}; \quad \alpha \neq 1$$

$$B(\alpha) = \int_0^{\frac{\pi}{2}} \cos x \sin(\alpha x)\, dx = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} (\sin(\alpha+1)x + \sin(\alpha-1)x)\right) dx$$

$$= -\frac{1}{2}\left[\frac{\cos(\alpha+1)x}{\alpha+1} + \frac{\cos(\alpha-1)x}{\alpha-1}\right]_0^{\pi/2}$$

$$= \frac{\alpha}{\alpha^2-1} - \frac{1}{2}\left[\frac{\cos(\alpha+1)\pi/2}{\alpha+1} + \frac{\cos(\alpha-1)\pi/2}{\alpha-1}\right]$$

$$= \frac{\alpha}{\alpha^2-1} - \frac{1}{2}\left[\frac{-\sin(\alpha\pi/2)}{\alpha+1} + \frac{\sin(\alpha\pi/2)}{\alpha-1}\right]$$

$$= \boxed{\frac{\alpha}{\alpha^2-1} - \frac{\sin(\alpha\pi/2)}{\alpha^2-1}}$$ 1

$$f(x) = \frac{f(x^+)+f(x^-)}{2}$$

$$= \frac{1}{\pi}\int_0^\infty \left(\frac{\cos(\alpha\pi/2)}{1-\alpha^2}\cos(\alpha x) + \left[\frac{\alpha}{\alpha^2-1} - \frac{\sin(\alpha\pi/2)}{\alpha^2-1}\right]\sin\alpha x\right) d\alpha; \quad \alpha \neq 1$$

At $x=0$

$$\frac{f(0^+)+f(0^-)}{2} = \frac{1}{2} = \frac{1}{\pi}\int_0^\infty \frac{\cos(\alpha\pi/2)}{1-\alpha^2}\, d\alpha; \quad \alpha \neq 1$$ 2

$$\pi/2 = \int_0^\infty \frac{\cos(\alpha\pi/2)}{1-\alpha^2}\, d\alpha; \quad \alpha \neq 1$$