KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
M204.TIME 3H, FULL MARKS 40, FINAL EXAM S1-2023/24

Question 1. [5,5,5] **a)** Find and sketch the largest region of the xy-plane for which the following initial value problem admits a unique solution

$$(x^2-4)\frac{dy}{dx}-\ln(y^2-3)=0,\ y(0)=2.$$

b) Solve the differential equation

$$(x^2+1)y\frac{dy}{dx}-y^2=1.$$

c) Solve the initial value problem

$$\begin{cases} 5xy^2\frac{dy}{dx}+y^3-(1+\ln x)y^{-2}=0, & x>0,\ y\neq 0 \\ y(1)=1. & \end{cases}$$

Question 2. [5] Use the undetermined coefficients method to find the general solution of the differential equation

$$y''-y'+y=1+e^x+\cos x.$$

Question 3. [5,5] **a)** Solve the differential equation

$$y''-2y'+y=x^{-2}e^x,\quad x>0.$$

b) Find at least the first four nonzero terms in a power series expansion about $x_0=0$ for the solution to the given initial value problem

$$y''+3xy'-y=0,\ \ y(0)=2,\ y'(0)=0.$$

Question 4. [5,5] **a)** Consider the function f defined by

$$f(x)=\begin{cases} 0 & \text{if } 0\leq x\leq \frac{\pi}{2} \\ x-\frac{\pi}{2} & \text{if } \frac{\pi}{2}\leq x<\pi \end{cases}.$$

Extend f as an odd-2π-periodic function, sketch the graph on the interval $(-3\pi,3\pi)$, find the Fourier series, and deduce that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\qquad \left(\text{Hint}: \sin\frac{n\pi}{2}=\begin{cases} 0 \text{ if } n=2k \\ (-1)^n \text{ if } n=2k+1\end{cases}\right)$$

b) Consider the function even :

$$f(x)=\begin{cases} 1, & 0\leq x\leq 1 \\ 2-x, & 1\leq x\leq 2 \\ 0, & x\geq 2 \end{cases}$$

Sketch the graph of f, find the Fourier integral representation of f, and deduce the value of the integral $\int_0^{\infty}\frac{1-\cos\lambda}{\lambda^2}d\lambda$.

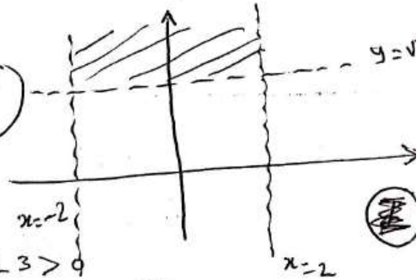
Q1. (a) Find and sketch the largest region of the xy-plane for which the initial value problem

$$\begin{cases} (x^2-4)\dfrac{dy}{dx} = \ln(y^2-3) \\ y(0) = 2 \end{cases}$$

has a unique solution.

Sol 1:

$$f(x,y) = \frac{\ln(y^2-3)}{x^2-4} \qquad (1)$$



$$\frac{\partial f}{\partial y} = \frac{2y}{(x^2-4)(y^2-3)}$$

f is cont. if $x \neq \pm 2$ and $y^2 - 3 > 0$

$\iff x \neq \pm 2$ and $|y| > \sqrt{3}$ (1)

$\iff x \neq \pm 2$ and $(y > \sqrt{3}$ or $y < -\sqrt{3})$ (1)

$\frac{\partial f}{\partial y}$ is cont. if $x \neq \pm 2$ and $y \neq \pm\sqrt{3}$

The largest region containing the point $(0,2)$ is (2)

$$R = \{(x,y) \in \mathbb{R}^2;\ -2 < x < 2,\ y > \sqrt{3}\}$$

(b): Solve the IVP
$$\begin{cases} (x^2+1)\,y\,\dfrac{dy}{dx} - y^2 = 1 \\ y(1) = 2 \end{cases}$$

Sol: $(x^2+1)\,y\,dy = (y^2+1)\,dx$ (1)

The D.E is separable

$$\frac{y}{y^2+1}\,dy = \frac{1}{x^2+1}\,dx$$ 2

$$\Rightarrow \int \frac{y}{y^2+1}\,dy = \int \frac{1}{x^2+1}\,dx$$

$$\Rightarrow \frac{1}{2}\ln(y^2+1) = \tan^{-1}x + c.$$ 2

(C) Solve the following I.V.P:

$$5xy^2y' + y^3 = (1+\ln x)\,y^{-2} \quad x>0,\quad y\neq 0$$

$$y(1)=1$$

Solution:

$$5y^2y' + \frac{1}{x}y^3 = \left(\frac{1+\ln x}{x}\right)y^{-2} \Rightarrow 5y^4y' + \frac{1}{x}y^5 = \frac{1+\ln x}{x} \quad (1)$$ 1

$$y^5 = u \Rightarrow 5y^4y' = u' \xrightarrow{\text{int. }(1)} u' + \frac{1}{x}u = \frac{1+\ln x}{x}$$

$$\mu(x) = e^{\int \frac{1}{x}dx} = x$$ 1

$$u = x^{-1}\left[\int x\left(\frac{1+\ln x}{x}\right)dx\right] + c\,x^{-1} = \frac{1}{x}\left[\int (1+\ln x)\,dx\right] + \frac{c}{x}$$

$$u = \frac{1}{x}\left[\cancel{x} + x\ln x - \cancel{x}\right] + \frac{c}{x} \Rightarrow u = \ln x + \frac{c}{x} \Rightarrow$$ 2

$$\left.\begin{array}{l} y = \left(\ln x + \frac{c}{x}\right)^{\frac{1}{5}} \\ y(1)=1 \Rightarrow 1 = (0+c)^{\frac{1}{5}} \Rightarrow c=1 \end{array}\right\} \Rightarrow y = \left(\ln x + \frac{1}{x}\right)^{\frac{1}{5}}$$ 1

3) Using Undetermined Coefficient Method to solve the differential equation

$$y'' - y' + y = 1 + e^x + \cos x$$

Ans. Characteristic eq $m^2 - m + 1 = 0$

$a = 1$; $b = -1$; $c = 1$

discriminant $\Delta = b^2 - 4ac = 1 - 4 = -3 = (\sqrt{3}\, i)^2$

The roots are: $m_1 = \frac{1 - \sqrt{3}\, i}{2}$; $m_2 = \frac{1 + \sqrt{3}\, i}{2}$ 1

The general solution for the homogeneous DE is

$$y_c(x) = c_1 e^{x/2} \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right)$$

- $f(x) = 1 + e^x + \cos x$, no repeating roots

So $y_p(x) = A + B e^x + C \cos x + D \sin x$ 2

$y'' - y' + y = 1 \Rightarrow y_{p_1} = 1$ So $A = 1$

$y'' - y' + y = e^x \Rightarrow y_{p_2} = e^x$ So $B = 1$

$y'' - y' + y = \cos x \Rightarrow y_{p_3} = -\sin x$ so $C = 0$ & $D = -1$ 1

We deduce that $y_p(x) = 1 + e^x - \sin x$

The general solution of DE: $y'' - y' + y = 1 + e^x + \cos x$

is $y(x) = y_c(x) + y_p(x) = c_1 e^{x/2} \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right) + 1 + e^x - \sin x$ 1

3) Using Un de ter...

Q3

لا يكتب في هذا الهامش

(a) $y'' - 2y' + y = \frac{e^x}{x^2}$

Homogeneous part

$y'' - 2y' + y = 0$, let $y = e^{mx}$ then

$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0$

$\Rightarrow y_c = c_1 e^x + c_2 x e^x$ (1)

let $y_1 = e^x$, $y_2 = xe^x$

To find y_p we have

$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x}$
$= e^{2x}$

$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x^2} & xe^x + e^x \end{vmatrix} = \frac{-e^{2x}}{x}$ (1)

$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x^2} \end{vmatrix} = \frac{e^{2x}}{x^2}$

$u_1' = \frac{W_1}{W} = -\frac{1}{x}$

$\Rightarrow u_1 = -\ln x$

$u_2' = \frac{W_2}{W} = \frac{1}{x^2} = x^{-2} \Rightarrow u_2 = \frac{-1}{x}$

$\Rightarrow y_p = u_1 y_1 + u_2 y_2 = (-\ln x) e^x + \frac{-1}{\cancel{x}} \cancel{x} e^x$ (2)

$= (-\ln x) e^x - e^x$

General solution

$y = y_c + y_p = c_1 e^x + c_2 x e^x - \ln x\, e^x - e^x$ (1)

Q3(b) Find at least the first four nonzero terms in a power series expansion about $x=0$ for the solution to the given initial value problem
$$\begin{cases} y'' + 3xy' - y = 0 \\ y(0) = 2, \; y'(0) = 0 \end{cases}$$

Ans

$y(x) = \sum_{n=0}^{\infty} a_n x^n$, Substitution into the diff eq:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \sum_{n=0}^{\infty} a_n x^n = 0 \quad \text{for } x \text{ near}$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \quad \text{for } x \text{ near}$$ 1

$$2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + 3n a_n - a_n\right]x^n = 0$$ 1

Therefore
$$\begin{cases} 2a_2 - a_0 = 0 \\ (n+2)(n+1)a_{n+2} + (3n-1)a_n = 0 \quad \text{for } n \geq 1 \end{cases}$$

Which leads to
$$\begin{cases} a_2 = \frac{1}{2}a_0 \\ a_{n+2} = \frac{1-3n}{(n+2)(n+1)}a_n \quad \text{for } n \geq 1 \end{cases}$$ 1

On the other hand, the initial values give
$$2 = y(0) = a_0 \; ; \; 0 = y'(0) = a_1$$

Therefore, we can compute successively:

$$a_2 = \tfrac{1}{2}a_0 = 1$$ 2
$$a_3 = -\tfrac{2}{6}a_1 = 0$$
$$a_4 = -\tfrac{5}{12}a_2 = -\tfrac{5}{12}$$
$$a_5 = -\tfrac{8}{20}a_3 = 0$$
$$a_6 = -\tfrac{11}{30}a_4 = \tfrac{11}{72}$$

Finally, $y(x) = 2 + x^2 - \frac{5}{12}x^4 + \frac{11}{72}x^6 + \cdots$

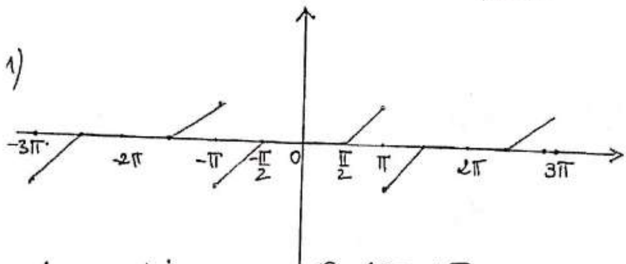
Q4) (a) Fourier series

Consider the odd-2π-periodic function defined by $f(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{\pi}{2} \\ x - \frac{\pi}{2} & \text{if } \frac{\pi}{2} \le x < \pi \end{cases}$

1) Sketch the graph of on $(-3\pi, 3\pi)$.
2) Find the Fourier of f.
3) Deduce that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$

Hint: $\sin \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n = 2k \\ (-1)^k & \text{if } n = 2k+1 \end{cases}$

1)



(1)

f is continuous on $\mathbb{R} \setminus (2\mathbb{Z}+1)\pi$.

2) f is odd; then $a_n = 0$; $\forall n \geqslant 0$.

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin(nt)\, dt = \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \left(t - \frac{\pi}{2}\right) \sin(nt)\, dt$$

$$= \frac{2}{\pi} \left(\left[\left(t - \frac{\pi}{2}\right) \cdot \left(-\frac{\cos nt}{n}\right) \right]_{\frac{\pi}{2}}^{\pi} + \frac{1}{n} \int_{\frac{\pi}{2}}^{\pi} \cos nt\, dt \right).$$

$$= \frac{2}{\pi} \left(-\frac{\pi}{2n} (-1)^n + \frac{1}{n^2} \left[\sin nt \right]_{\frac{\pi}{2}}^{\pi} \right)$$

$$= \frac{2}{\pi}\left(\frac{-\pi}{2n}(-1)^n - \frac{1}{n^2}\sin n.\frac{\pi}{2}\right)$$
$$= \frac{-2}{\pi}\left(\frac{\pi}{2n}(-1)^n + \frac{1}{n^2}\sin n.\frac{\pi}{2}\right).$$

$$\begin{cases} \cdot\ b_{2n} = \frac{-2}{\pi}\cdot\frac{\pi}{4n} = \frac{-1}{2n}. \\ \cdot\ b_{2n+1} = \frac{-2}{\pi}\left(\frac{-\pi}{2(2n+1)} + \frac{1}{(2n+1)^2}\cdot(-1)^n\right) \\ \qquad = \frac{1}{2n+1} - \frac{2}{\pi}\cdot\frac{(-1)^n}{(2n+1)^2}. \end{cases}$$

2

Hence
$$FS(f,x) = \sum_{n\geq 1}^{\infty} b_n \sin(nx)$$
$$= \frac{-2}{\pi}\sum_{n\geq 1}\left(\frac{\pi}{2n}(-1)^n + \frac{1}{n^2}\sin\left(n\frac{\pi}{2}\right)\right)\sin(nx)$$
$$= f(x) \quad ; \quad \forall x \in \mathbb{R}\setminus(2\mathbb{Z}+1)\pi.$$

3) For $x = \frac{\pi}{2}$.
$$FS(f,x) = \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{2}\right). = f\left(\frac{\pi}{2}\right).$$

2

$$\Rightarrow \sum_{n=0}^{\infty} b_{2n+1}\sin(2n+1)\frac{\pi}{2} = \sum_{n=0}^{\infty}(-1)^n\cdot b_{2n+1}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 0$$
$$\Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)} \sim$$

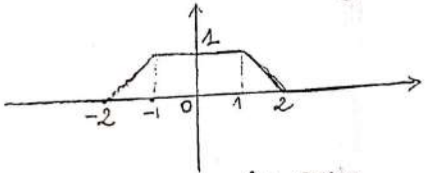
(b) Fourier integral. (3

Consider the even function f defined by
$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } 1 \le x \le 2. \\ 0 & \text{if } x \geq 2 \end{cases}$$

1) Sketch the graph f
2) Find the Fourier integral of f
3) Deduce the value of $\int_0^{\infty} \frac{1-\cos\lambda}{\lambda^2} d\lambda$.

Solution.

1)



1

2) $B(\lambda) = 0$; since f is even.

$$A(\lambda) = 2 \int_0^{\infty} f(t) \cos\lambda t \, dt$$
$$= 2\left(\int_0^1 \cos\lambda t \, dt + \int_1^2 (2-t)\cos\lambda t \, dt \right)$$
$$= 2\left(\left[\frac{\sin\lambda t}{\lambda}\right]_0^1 + \left[\frac{(2-t)\sin\lambda t}{\lambda}\right]_1^2 + \int_1^2 \frac{\sin\lambda t}{\lambda} dt \right) \quad 2$$
$$= 2\left(\frac{\sin\lambda}{\lambda} - \frac{\sin\lambda}{\lambda} - \left[\frac{\cos\lambda t}{\lambda^2}\right]_1^2 \right) = \frac{2(\cos\lambda - \cos 2\lambda)}{\lambda^2}$$

Then $FS(f,x) = \frac{1}{\pi}\left(\int_0^{+\infty} A(\lambda)\cos\lambda x \, d\lambda + \int_0^{\infty} B(\lambda)\sin\lambda x \, d\lambda \right)$
$$= \frac{1}{\pi} \int_0^{\infty} \frac{2(\cos\lambda - \cos 2\lambda)}{\lambda^2} \cdot \cos\lambda x \, d\lambda = f(x) \; ; \; \forall x \in \mathbb{R}$$

3) $\frac{2}{\pi}\int_0^\infty \left(\frac{\cos\lambda - \cos 2\lambda}{\lambda^2}\right)\cos\lambda x \, d\lambda = f(x); \ \forall x\in\mathbb{R}$ (4

If $x=0$; $\frac{2}{\pi}\int_0^\infty \frac{\cos\lambda - \cos 2\lambda}{\lambda^2} \cdot d\lambda = f(0) = 1$ 1

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda - 1}{\lambda^2} - \frac{2}{\pi}\int_0^\infty \frac{\cos 2\lambda - 1}{\lambda^2}\, d\lambda = 1.$$

But, $\int_0^\infty \frac{(\cos 2\lambda - 1)}{\lambda^2}\, d\lambda = 2\int_0^\infty \left(\frac{\cos\lambda - 1}{\lambda^2}\right) d\lambda$

then. $\frac{2}{\pi}\cdot\left(\int_0^\infty \frac{\cos\lambda - 1}{\lambda^2}\, d\lambda - 2\int_0^\infty \frac{\cos\lambda - 1}{\lambda^2}\, d\lambda\right) = 1$

$$\Rightarrow \int_0^\infty \frac{1 - \cos\lambda}{\lambda^2}\, d\lambda = \frac{\pi}{2}. \quad \#$$ 1

KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
M204.TIME 3H, FULL MARKS 40, FINAL EXAM S1-2023/24

Question 1. [5,5,5] **a)** Find and sketch the largest region of the xy-plane for which the following initial value problem admits a unique solution

$$(x^2-4)\frac{dy}{dx}-\ln(y^2-3)=0,\ y(0)=2.$$

b) Solve the differential equation

$$(x^2+1)y\frac{dy}{dx}-y^2=1.$$

c) Solve the initial value problem

$$\begin{cases} 5xy^2\frac{dy}{dx}+y^3-(1+\ln x)y^{-2}=0, & x>0,\ y\neq 0\\ y(1)=1. \end{cases}$$

Question 2. [5] Use the undetermined coefficients method to find the general solution of the differential equation

$$y''-y'+y=1+e^x+\cos x.$$

Question 3. [5,5] **a)** Solve the differential equation

$$y''-2y'+y=x^{-2}e^x,\quad x>0.$$

b) Find at least the first four nonzero terms in a power series expansion about $x_0=0$ for the solution to the given initial value problem

$$y''+3xy'-y=0,\quad y(0)=2,\ y'(0)=0.$$

Question 4. [5,5] **a)** Consider the function f defined by

$$f(x)=\begin{cases} 0 & \text{if } 0\leq x\leq \frac{\pi}{2}\\ x-\frac{\pi}{2} & \text{if } \frac{\pi}{2}\leq x<\pi \end{cases}.$$

Extend f as an odd-2π-periodic function, sketch the graph on the interval $(-3\pi,3\pi)$, find the Fourier series, and deduce that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\qquad \left(\text{Hint}: \sin\frac{n\pi}{2}=\begin{cases} 0 \text{ if } n=2k\\ (-1)^n \text{ if } n=2k+1 \end{cases}\right)$$

b) Consider the function even :

$$f(x)=\begin{cases} 1, & 0\leq x\leq 1\\ 2-x, & 1\leq x\leq 2\\ 0, & x\geq 2 \end{cases}$$

Sketch the graph of f, find the Fourier integral representation of f, and deduce the value of the integral $\int_0^{\infty}\frac{1-\cos\lambda}{\lambda^2}d\lambda$.