KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
TIME: 3H, FULL MARKS: 40, SI /29/04/1441 MATH 204

Question 1. [4,4] **a)** A boy with a thermometer in his pocket reading 40^0C falls in a swimming pool whose temperature is maintained at 30^0C. If after 1 minute the thermometer reads 32^0C, what will be the reading after 3 minutes.

b) Find the general solution of the differential equation

$$(4x\sin y + 6)dx + (x^2\cos y)dy = 0, \quad x > 0.$$

Question 2. a) [4,5]. Solve the initial value problem

$$y' = \frac{(y-2x+1)^2}{y-2x}, \quad y(0) = 4\sqrt{3}.$$

b) Find an interval I for which the following initial value problem has a unique solution

$$(4-x^2)y'' + \frac{x}{\sqrt{x+1}}y' + y\ln(1-\frac{x}{4}) = 0, y(0) = 1, y'(0) = 0.$$

Question 3. a) [4,4]. Use undetermined coefficients method to solve the differential equation

$$y'' - y' - 2y = 4e^{3x} + 5\sin x.$$

b) Solve the differential equation

$$y'' - 6y' + 9y = \frac{e^{3x}}{1+x}.$$

Question 4 [5]. Use power series method to find the power series solution about the ordinary point $x_0 = 0$ for the differential equation

$$(x-1)y'' - xy' + y = 0.$$

Question 5. a) [5,5]. Let $f(x)$ be a $2\pi-$periodic function defined by:

$$f(x) = \begin{cases} 1, & -\pi < x < -\frac{\pi}{2}, \ \frac{\pi}{2} < x < \pi \\ 0, & |x| \leq \frac{\pi}{2}. \end{cases}$$

Sketch the graph of f, find its Fourier Series, and deduce that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

b) Consider the function

$$f(x) = \begin{cases} x, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Sketch the graph of f, find its Fourier integral and deduce the value of the integral

$$\int_0^{\infty} \frac{2\sin^2\lambda - \lambda\sin(2\lambda)}{\lambda^2}d\lambda.$$

Complete Solution of final Exam.

M. 204. First Semester 1441

Question 1

(a) $\dfrac{dT/dt}{T-T_s} = k$, $\dfrac{dT}{T-T_s} = k\,dt$ (1)

$\ln|T-T_s| = kt+c$, $T-T_s = c_1 e^{kt}$; $c_1 = \mp e^{c} \neq 0$

$T(0) = 40$, $T_s = 30$, $T(t) = 30 + c_1 e^{kt}$

$T(0) = 40 = 30 + c_1 \Rightarrow c_1 = 10$

$T(t) = 30 + 10e^{kt}$, $T(1) = 32 = 30 + 10e^{k}$, (2)

$0.2 = \frac{1}{5} = e^{k}$, then $\boxed{T(t) = 30 + 10(0.2)^t}$ (1)

For $t=3$, we have $T(3) = 30 + 10(0.2)^3$

$T(3) = 30 + 0.08 = \boxed{30.08\ C^\circ}$

Question 2

(a) $y' = \dfrac{dy}{dx} = \dfrac{(y-2x+1)^2}{y-2x}$; $y(0) = 4\sqrt{3}$; $y \neq (2x)$

we put $u = y-2x$, $u' = y'-2$ or $y' = u'+2$ (1)

$u'+2 = \dfrac{(u+1)^2}{u} = \dfrac{u^2+2u+1}{u}$

$u' + \cancel{2} = u + \cancel{2} + \frac{1}{u}$

$\dfrac{du}{dx} = u' = \dfrac{u^2+1}{u} \Rightarrow \dfrac{u\,du}{u^2+1} = dx \Rightarrow \frac{1}{2}\ln(u^2+1) = x + c$

$\boxed{\ln(u^2+1) = 2x + c_1}$ $(c_1 = 2c)$

or $\ln\left((y-2x)^2+1\right) = 2x + c_1$ is the solution of the D.E. (2)

But $y(0) = 4\sqrt{3} \Rightarrow \ln\left((4\sqrt{3})^2+1\right) = c_1$, $\ln 49 = c_1$ or $\boxed{2\ln 7 = c_1}$

Hence $\boxed{\ln\left((y-2x)^2+1\right) = 2x + 2\ln 7}$ is the solution of the IVP (1)

Question 1 (b) $(4x\sin y+6)dx + \underbrace{(x^2\cos y)}_{N}dy = 0$; $x>0$

$-\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\frac{\partial M}{\partial y} = 4x\cos y , \qquad \frac{\partial N}{\partial x} = 2x\cos y$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4x\cos y - 2x\cos y}{x^2\cos y} = \frac{2}{x}$$

$\mu(x) = e^{\int \frac{2}{x}dx} = e^{\ln x^2} = x^2$, is an integrating factor 1

Then $\underbrace{(4x^3\sin y + 6x^2)}_{M}dx + \underbrace{(x^4\cos y)}_{N}dy = 0$ is an exact equation

because

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x^3\cos y$$

So $\exists$ a function F of x and y s.t

$$\frac{\partial F}{\partial x} = M = 4x^3\sin y + 6x^2 , \qquad \frac{\partial F}{\partial y} = x^4\cos y \qquad 1$$

$$F(x,y) = \int x^4\cos y\, dy = x^4\sin y + \phi(x)$$

$$\frac{\partial F}{\partial x} = 4x^3\sin y + \phi'(x) = 4x^3\sin y + 6x^2 \qquad 2$$

$$\phi'(x) = 6x^2 \Rightarrow \phi(x) = 2x^3 + c$$

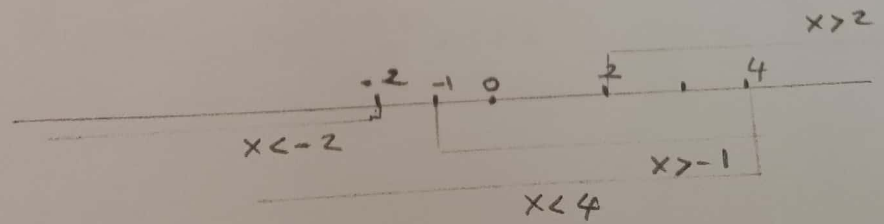
Then $F(x,y) = x^4\sin y + 2x^3 + c = 0$ is the Solution of the D.E

Question 2 (b)
$$\begin{cases} (4-x^2)y'' + \dfrac{x}{\sqrt{x+1}}y' + \ln\left(1-\dfrac{x}{4}\right)y = 0 \\ y(0)=1 , \quad y'(0)=0 \end{cases}$$

$a_2(x) = (4-x^2)$, is continuous on $\mathbb{R}$ and $a_2(x) \neq 0$ if $x \neq \mp 2$

$a_1(x) = \dfrac{x}{\sqrt{x+1}}$ is continuous on $(-1, \infty)$ 2

$g(x) = \ln\left(1-\dfrac{x}{4}\right)$ is continuous on $1-\dfrac{x}{4} > 0$ or $x<4$



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Then a_2, a_1 and a_0, with $x \neq \mp 2$, are continuous on

$D = \{x \in \mathbb{R} ; -1 < x < 2\} \cup \{x \in \mathbb{R}, 2 < x < 4\}$. But as $0 \in (-1, 2)$, then the longest interval for which the IVP has a unique solution is $\boxed{I = (-1, 2)}$ 2

Question 3 (a) $y'' - y' - 2y = 4e^{3x} + 5\sin x$

1) $y'' - y' - 2y = 0$, $y = e^{mx}$; $m^2 - m - 2 = (m-2)(m+1) = 0$

$m = 2$, $m = -1$

$y_c = c_1 e^{-x} + c_2 e^{2x}$ 1

2) $y_p = Ae^{3x} + B\sin x + C\cos x$ 1

$y'_p = 3Ae^{3x} + B\cos x - C\sin x$, $y''_p = 9Ae^{3x} - B\sin x - C\cos x$

$y''_p - y'_p - 2y_p = 9Ae^{3x} - B\sin x - C\cos x - 3Ae^{3x} - B\cos x + C\sin x$
$- 2Ae^{3x} - 2B\sin x - 2C\cos x = 4e^{3x} + 5\sin x$

$4Ae^{3x} + (-3B + C)\sin x + (-3C - B)\cos x = 4e^{3x} + 5\sin x$

Then $A = 1$, $-3B + C = 5$
$-3C - B = 0$ $\Rightarrow$ $A = 1$, $B = -\frac{3}{2}$, $C = \frac{1}{2}$ 2

So $\boxed{y_p = e^{3x} - \frac{3}{2}\sin x + \frac{1}{2}\cos x}$ and $y = y_c + y_p$ is the general solution of the D.E.

(b) $y'' - 6y' + 9y = \dfrac{e^{3x}}{1+x}$; $x > -1$

1) $y'' - 6y' + 9y = 0$, $y = e^{mx}$, $m^2 - 6m + 9 = (m-3)^2 = 0$

$m_1 = m_2 = m = 3$

$y_c = c_1 e^{3x} + c_2 xe^{3x}$, $y_1 = e^{3x}$, $y_2 = xe^{3x}$ 1

2) $W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$, $y_p = u_1 y_1 + u_2 y_2$
$= u_1 e^{3x} + xe^{3x} u_2$

$$W_1 = \begin{vmatrix} 0 & xe^{3x} \\ \frac{e^{3x}}{1+x} & (3x+1)e^{3x} \end{vmatrix} = \frac{-xe^{6x}}{1+x}$$

$$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & \frac{e^{3x}}{1+x} \end{vmatrix} = \frac{e^{6x}}{1+x},$$

1

$$u_1' = \frac{W_1}{W} = \frac{-x}{1+x} \Rightarrow u_1 = \ln(1+x) - x$$

$$u_2' = \frac{W_2}{W} = \frac{1}{1+x} \Rightarrow u_2 = \ln(1+x), \text{ then}$$

$$\boxed{y_p = e^{3x}(\ln(1+x) - x) + xe^{3x}\ln(1+x)}$$

2

Thus, we get the general solution of the D.E.

$$\boxed{y = y_c + y_p = c_1 e^{3x} + c_2 x e^{3x} + e^{3x}(\ln(1+x) - x) + xe^{3x}\ln(1+x)}$$

Question: 4 $(x-1)y'' - xy' + y = 0$ about $x_0 = 0$

$$\frac{a_1}{a_2} = -\frac{x}{x-1} = \frac{x}{1-x} = \sum_0^\infty x^{n+1} \quad ; \ |x|<1$$

$$\frac{a_0}{a_2} = \frac{-1}{1-x} = -\sum_0^\infty x^n, \quad |x|<1 \quad , \quad y = \sum_0^\infty a_n x^n$$

$$(x-1)\sum_2^\infty n(n-1)a_n x^{n-2} - x\sum_1^\infty n a_n x^{n-1} + \sum_0^\infty a_n x^n = 0$$

1

$$\sum_2^\infty n(n-1)a_n x^{n-1} - \sum_2^\infty n(n-1)a_n x^{n-2} - \sum_1^\infty n a_n x^n + \sum_0^\infty a_n x^n = 0$$

$n-1 = k$, $n = k+1$ | $n-2 = k$, $n = k+2$ | $n = k$ | $n = k$

$$\sum_1^\infty k(k+1)a_{k+1} x^k - \sum_0^\infty (k+1)(k+2)a_{k+2} x^k - \sum_1^\infty k a_k x^k + \sum_0^\infty a_k x^k = 0$$

$$(-2a_2 + a_0) + \sum_1^\infty \left[k(k+1)a_{k+1} - (k+1)(k+2)a_{k+2} - k a_k + a_k \right] x^k = 0$$

Then: $-2a_2 + a_0 = 0 \Rightarrow a_2 = \frac{1}{2} a_0$

$$(k+1)(k+2)\, a_{k+2} = k(k+1) a_{k+1} - (k-1) a_k$$

$$a_{k+2} = \frac{k(k+1) a_{k+1} - (k-1) a_k}{(k+1)(k+2)}, \quad k \geq 1 \quad (2)$$

$k=1 \Rightarrow a_3 = \frac{1}{6}(2a_2 - 0) = \frac{1}{3} a_2 = \frac{1}{6} a_0$

$k=2 \Rightarrow a_4 = \frac{1}{12}(6a_3 - a_2) = \frac{1}{12}\left(a_0 - \frac{1}{2}a_0\right) = \frac{1}{24} a_0, \ldots$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y = a_1 x + a_0 \left[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\right] \quad |x| < 1, \quad (2)$$

$= a_0 y_1 + a_1 y_2$, where $y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$

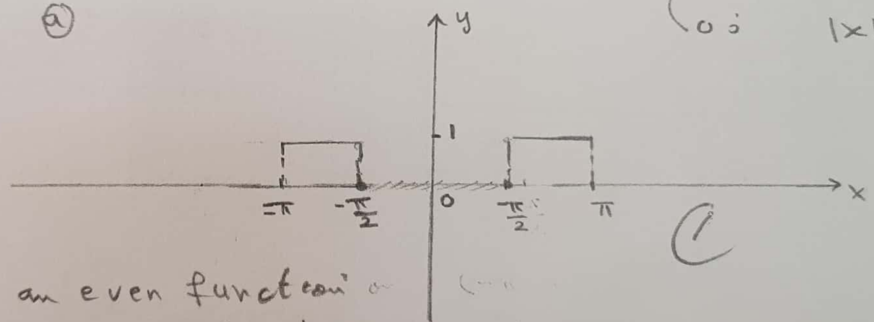
is the solution of the D.E. $\quad y_2 = x$

Question 5:

$$f(x) = \begin{cases} 1; & -\pi < x < -\frac{\pi}{2}, \\ & \frac{\pi}{2} < x < \pi \\ 0; & |x| \leq \frac{\pi}{2} \end{cases}$$

(a)

$T = \pi$



(1)

f is an even function on $(-\pi, \pi)$, then $b_n = 0$, $n = 1, 2, \ldots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{\pi/2}^{\pi} (1)\,dx = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1 \quad (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos nx\,dx = \frac{2}{n\pi}\left[\sin(nx)\right]_{\pi/2}^{\pi} \quad (1)$$

$$= \frac{2}{n\pi}\left[\sin n\pi - \sin\left(\frac{n\pi}{2}\right)\right] = \boxed{\frac{2}{n\pi}\left(-\sin\frac{n\pi}{2}\right)}$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0; & n = 2m \\ (-1)^{n-1}; & n = 2m-1 \\ & m \geq 1 \end{cases}$$

$$\frac{f(x^+)+f(x^-)}{2} = \frac{a_0}{2} + \sum_1^\infty a_n \cos(nx)$$

$$\frac{f(x^+)+f(x^-)}{2} = 1/2 + \sum_1^\infty \frac{-2}{n\pi} \sin\left(n\frac{\pi}{2}\right) \cos(nx)$$

$$\frac{f(x^+)+f(x^-)}{2} = 1/2 + \sum_1^\infty \frac{-2}{(2n-1)\pi} (-1)^{n-1} \cos((2n-1)x)$$

$$\boxed{\frac{f(x^+)+f(x^-)}{2} = 1/2 + \sum_1^\infty \frac{2(-1)^n}{(2n-1)\pi} \cos((2n-1)x) \ ; \ -\pi < x < \pi}$$

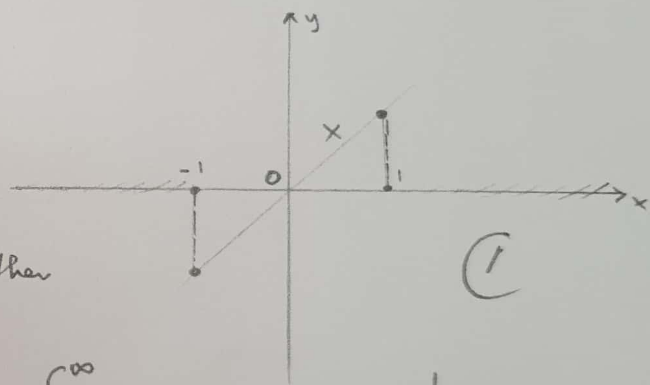
$$\frac{f(x^+)+f(x^-)}{2} = 1/2 + \sum_0^\infty \frac{2(-1)^{n+1}}{(2n+1)\pi} \cos((2n+1)x) \ ; \ -\pi < x < \pi$$

At $x=0$ $\quad f(0) = \frac{f(0^+)+f(0^-)}{2} = 0 = \frac{1}{2} + \frac{2}{\pi}\sum_0^\infty \frac{(-1)^n(-1)}{(2n+1)}$ 2

Then $\boxed{\sum_0^\infty \frac{(-1)^n}{2n+1} = \frac{\pi}{4}}$ or $\sum_1^\infty \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

(b)

$$f(x) = \begin{cases} x \ ; & |x| \le 1 \\ 0 \ ; & |x| > 1 \end{cases}$$



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f is an odd function on $\mathbb{R}$, then

$A(\alpha) = 0,$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin(\alpha x)\, dx = 2\int_0^\infty f(x) \sin \alpha x\, dx = 2\int_0^1 x \sin \alpha x\, dx$$

$$B(\alpha) = 2\left[x\left(-\frac{\cos \alpha x}{\alpha}\right)\right]_0^1 + 2\int_0^1 \frac{\cos(\alpha x)}{\alpha} dx = -\frac{2\cos\alpha}{\alpha} + \left[\frac{2}{\alpha^2}\sin \alpha x\right]_0^1$$

$$\boxed{B(\alpha) = \frac{-2\cos\alpha}{\alpha} + \frac{2}{\alpha^2}\sin\alpha}$$ 2

$$\frac{f(x^+)+f(x^-)}{2} = \frac{1}{\pi}\int_0^\infty \left(\frac{-2\cos\alpha}{\alpha} + \frac{2\sin\alpha}{\alpha^2}\right) \sin \alpha x\, d\alpha \ , \ x \in \mathbb{R}$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2} \sin \alpha x\, d\alpha \ , \ x \in \mathbb{R}$$

At $x=1$, we have

$$\frac{f(1^+)+f(1^-)}{2} = \frac{1+0}{2} = \frac{1}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha-\alpha\cos\alpha}{\alpha^2}\sin\alpha \, d\alpha$$

$$\frac{1}{2} = \frac{1}{\pi}\int_0^\infty \frac{2\sin^2\alpha-2\alpha\sin\alpha\cos\alpha}{\alpha^2}\, d\alpha$$

$$\boxed{\frac{\pi}{2} = \int_0^\infty \frac{2\sin^2\alpha-\alpha\sin(2\alpha)}{\alpha^2}\, d\alpha}$$

(2)