إعداد :
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Department of Mathematics
King Saud University
Final Examination (M-203)
(Summer Semester)[(1430/1431]

Max. Marks: 50

Time: 180 minutes

Marking Scheme: Q.No:1[3+5+3], Q.No:2[5+4+4], Q.No:3[4+4+4], Q.No:4[5+5+4]

Q. No: 1 (a) Determine whether the sequence $\left\{\dfrac{5^n-2^n}{3^n}\right\}_{n=1}^{n=\infty}$ converges or diverges and if it converges, find its limit.

(b) Find the radius of convergence and interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(2x+1)^n}{n^2+1}$.

(c) Use a power series representation $\dfrac{1}{1-x}=\displaystyle\sum_{n=0}^{\infty}x^n, |x|<1$ to obtain power Series representation.

for the function $f(x)=\dfrac{3}{5-x}$ centered at . c = 1.

Q. No: 2 (a) Find first four terms of the Taylor series for $f(x)=\sin x$ at $c=\dfrac{\pi}{4}$.

(b) Reverse the order of integration and evaluate $\displaystyle\int_0^2\int_{x^2}^4\sqrt{y}\sin(y)\,dydx$.

(c) Find the surface area of the solid bounded by the surface $z=x^2+y^2-9$ and below the xy-plane.

Q. No: 3 (a) Use **spherical co-ordinates** to find the volume of the solid that lies above the **Cone** $z=\sqrt{x^2+y^2}$ and below the **sphere** $x^2+y^2+z^2=9$.

(b) Use **cylindrical co-ordinates** to evaluate the integral $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}dzdydx$.

(c) Evaluate the line integral $\displaystyle\int_C xdx+ydy-5zdz$, where the curve C is given parametrically by the

equations $C: x=2\cos t, y=2\sin t, z=t, 0\le t\le 2\pi$.

Q. No: 4 (a) Use **Green's theorem** to evaluate the line integral $\displaystyle\oint_C y^4dx+(x^4+4xy^3)dy$ where C is the path from

(0,0) to (1,1) along the graph of $y=x^3$ and from (1,1) to (0,0) along the graph of $y=x$.

(b) Use the **Divergence theorem** to evaluate the integral $\displaystyle\iint_S\vec{F}\cdot\vec{n}\,dS$ where $\vec{F}=x^3\,\vec{i}+y^3\,\vec{j}+0\,\vec{k}$ and S

is the surface of the region bounded by $z=3-x^2-y^2$ and $z=-1$.

(c) If $\vec{F}=x^2\,\vec{i}+y^2\,\vec{j}+z^2\,\vec{k}$ and C is the curve of intersection of the plane $z+x=0$ and the cylinder

$x^2+y^2=1$, **use Stokes's** theorem to evaluate $\displaystyle\oint_C\vec{F}\cdot d\vec{r}$.

1

1 (a) $a_n = (5/3)^n - (2/3)^n \longrightarrow \infty$ dg't

$\downarrow \infty \quad \downarrow 0$

(b) $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(2x+1)^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{(2x+1)^n}\right|$

$= \frac{n^2+1}{n^2+2n+2}|2x+1| \longrightarrow |2x+1|$

Absolutely c'gt if $|2x+1| < 1$

$-1 < 2x+1 < 1$

$-2 < 2x < 0 \Longrightarrow \boxed{-1 \leq x < 0}$

$\boxed{\text{At } x=-1}$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ By AST c'gt

$\boxed{\text{At } x=0}$ $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ by Comparison test c'gt

Interval of c'gence $[-1, 0]$

radius " " $\rho = \frac{1}{2}$

Q. 1. (c)

$$f(x) = \frac{3}{5-x} = \frac{3}{5-(x-1+1)}$$

$$= \frac{3}{5-(x-1)-1} = \frac{3}{4-(x-1)}$$

$$= \frac{3}{4}\left[\frac{1}{1-\left(\frac{x-1}{4}\right)}\right]$$

If $\left|\frac{x-1}{4}\right| < 1$ then

$$f(x) = \frac{3}{4}\left[1 + \left(\frac{x-1}{4}\right) + \left(\frac{x-1}{2}\right)^2 + \cdots\cdots\right]$$

Q. 2 (a)

$$f(c) + (x-c)\,f'(c) + \frac{(x-c)^2}{2!}f''(c) + \frac{(x-c)^3}{3!}f'''(c)$$

$$f(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$$

$$f'(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$$

$$f''(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}$$

$$f'''(\pi/4) = -\cos(\pi/4) = -1/\sqrt{2}$$

Taylor Series

$$\frac{1}{\sqrt{2}} + (x-\pi/4)\left(\frac{1}{\sqrt{2}}\right) - \frac{(x-\pi/4)^2}{2!}(1/\sqrt{2})$$

$$- \frac{(x-\pi/4)^3}{3!}(1/\sqrt{2}$$

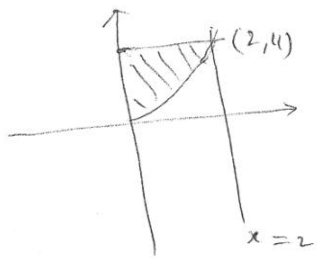
③

Q, 2 (b) $\displaystyle\int_0^2 \int_{x^2}^4 \sqrt{y}\,\sin(y)\,dy\,dx$

$0 \le x \le 2$

$x^2 \le y \le 4$

$$\boxed{\begin{array}{l} 0 \le y \le 4 \\ 0 \le x \le \sqrt{y} \end{array}}$$



$$= \int_0^4 \int_0^{\sqrt{y}} \sqrt{y}\,\sin y\,dx\,dy$$

$$= \int_0^4 \sqrt{y}\,\sin y\,\big[x\big]_0^{\sqrt{y}}\,dy$$

$$= \int_0^4 y\sin y\,dy = y(-\cos y) + \int \cos y\,dy\Big]_0^4$$

$$= -y\cos y + \sin y\Big]_0^4$$

$$= -4\cos(4) + \sin(4) - 0$$

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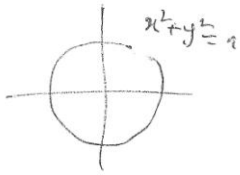
Q: 2 (c)

$$S.A = \iint_{R_{xy}} \sqrt{1 + f_x^2 + f_y^2}\; dA$$

$$z = f(x,y) = x^2 + y^2 - 9$$



$$f_x = 2x, \quad f_y = 2y$$

$$S.A = \iint_{R_{xy}} \sqrt{1 + 4x^2 + 4y^2}\; dA = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\; r\,dr\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}\int_0^3 (1+4r^2)^{1/2}(8r)\,dr\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \left.\frac{(1+4r^2)^{3/2}}{3/2}\right]_0^3 d\theta$$

$$= \frac{1}{12}\left[(14)^{3/2} - 1\right]\int_0^{2\pi} d\theta = \frac{1}{12}\left(14^{3/2} - 1\right)(2\pi)$$

$$= \frac{1}{6}\left(14^{3/2} - 1\right)\pi$$

//

(5)

(#32d) Use cylindrical coordinates to evaluate

[Mark: 4]

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx \qquad 1+1+1 = (3)$$

Soln. $\displaystyle\int_0^{\pi}\int_0^{2}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

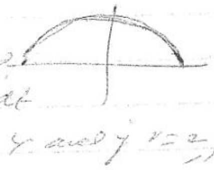
$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$y^2 = 4 - x^2$

or, $x^2 + y^2 = 4$

$z = \sqrt{4-r^2}$

$$= \int_0^{\pi}\int_0^{2} r\sqrt{4-r^2}\,dr\,d\theta$$



$$= \frac{8}{3}\times\pi \qquad (1)$$

Put $4 - r^2 = t$

$-2r\,dr = dt$

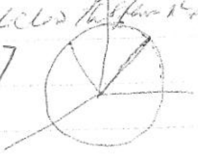
If $r = 0$, $t = 4$ and if $r = 2$, $t = 0$

$$-\frac{1}{2}\int_4^0 \sqrt{t}\,dt = \frac{1}{2}\left[\frac{2}{3}t^{3/2}\right]_0^4$$

$$= \frac{1}{2}\left[\frac{2}{3}(4)^{3/2}\right] = \frac{8}{3}$$

32c) Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 9$

[Mark: 4]



$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} \rho^2 \sin\phi \,d\rho\,d\phi\,d\theta \qquad 1+1+1 = (3)$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \sin\phi\left[\frac{\rho^3}{3}\right]_0^3 d\phi\,d\theta$$

$$= \frac{9}{2}\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/4} d\theta = \frac{9}{2}\left(-\frac{1}{\sqrt{2}}+1\right)2\pi = \frac{9}{2}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)2\pi$$

(1) $= \cancel{\ldots}\ 9\pi$

$= 9\sqrt{2}(\sqrt{2}-1)$

$\simeq 12.727(.414$

$\simeq 5.2719(\pi)$

$1 - \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2}-1}{\sqrt{2}}$

$\sqrt{2} = 1.4142$

(6)

Q: 3(c) $\int_C x\,dx + y\,dy - 5z\,dz$ [Mark: 4]

C: $x = 2\cos t$, $y = 2\sin t$, $z = t$, $0 \le t \le 2\pi$

$dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$, $dz = dt$

(1)

$= \int_0^{2\pi} (2\cos t)(-2\sin t\,dt) + (2\sin t)(2\cos t\,dt) - 5t\,dt$

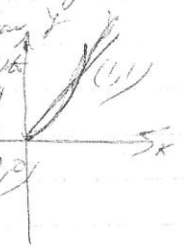
(2)

$= -5\int_0^{2\pi} t\,dt = -5\left[\frac{t^2}{2}\right]_0^{2\pi} = -\frac{5(4\pi^2)}{2}$

$= -10\pi^2$ (1)

7

Q#4(a) Use Green's theorem to evaluate the line integral $\oint_C y^4\,dx + (x^4 + 4xy^3)\,dy$ where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$ [Marks:5]



$$\iint_R (4x^3 + \cancel{4y^3} - \cancel{4y^3})\,dA$$

$$= 4\int_0^1\int_{x^3}^{x} x^3\,dy\,dx \qquad 2+2=4$$

$$= 4\int_0^1 x^3 [y]_{x^3}^{x}\,dx = 4\int_0^1 x^3(x - x^3)\,dx$$

$$= 4\int_0^1 (x^4 - x^6)\,dx$$

$$= 4\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = 4\left[\frac{1}{5} - \frac{1}{7}\right]$$

$$= 4\left[\frac{7-5}{35}\right]$$

1

$$= \frac{8}{35}$$

4(b) Use the divergence theorem to evaluate the integral $\iint_S \vec{F}\cdot\vec{n}\,dS$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + 0\vec{k}$ and S is the surface of the region bounded by $z = 3 - x^2 - y^2$ and $z = -1$. [Marks:5]

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_Q (3x^2 + 3y^2)\,dV \quad (1)$$

$$= \int_0^{2\pi}\int_0^2\int_{-1}^{3-r^2} 3r^2\, r\,dz\,dr\,d\theta \quad (3)$$

$$= 3\int_0^{2\pi}\int_0^2 r^3\left[3 - r^2 + 1\right]dr\,d\theta$$

$$= 3\int_0^{2\pi}\left[4\frac{r^4}{4} - \frac{r^6}{6}\right]_0^2 d\theta$$

$$= 3(2\pi)\left(16 - \frac{64}{6}\right)$$

$$= 6\pi\left(\frac{96-64}{6}\right) = \cancel{6}\pi\left(\frac{32}{\cancel{6}}\right) = 32\pi$$

1

(8)

Q #4(c) If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and C is the curve of the intersection of the plane $z + x = 0$ and the cylinder $x^2 + y^2 = 1$ use Stoke's theorem to evaluate $\oint_C \vec{F} \cdot d\vec{r}$ [Mark: 4]

$$\iint \vec{F} \cdot d\vec{r} = \iint_{R_{xy}} (-M_1 g_x - N_1 g_y + P_1)\, dA = \underline{\underline{0}}$$

$$= \iint_S (\text{curl}\,\vec{F}) \cdot \vec{k}\, ds = \iint$$

$z = -x = g$

$g_x = -1$ and $g_y = 0$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0) + \vec{k}(0)$$

$$= M_1\vec{i} + N_1\vec{j} + P_1\vec{k}$$