MID TERM II EXAMINATION
SEMESTER I, 1433-1434
Department of Mathematics
King Saud University
MATH: 203 Time: 90 Minutes Full Marks: 25

Question # 1. [Marks: 4]

Evaluate the integral $\int_0^2 \int_y^2 e^{x^2}\, dxdy$.

Question # 2. [Marks: 4]

Evaluate the integral using polar coordinates.

$$\int\int_R \sqrt{4-x^2-y^2}\, dA,$$

where R is the region bounded by the circle $x^2+(y-1)^2=1$.

Question # 3. [Marks: 4]

Find the surface area of the portion of the surface $z=xy$ that is above the region in the first quadrant bounded by the lines $y=x, y=0$, and the circle $x^2+y^2=9$.

Question # 4. [Marks: 4]

The area mass density of the lamina in the shape of the region bounded by the graphs of $y^2=x$ and $x+y=2$ is given by $\delta(x,y)=x$. Find the mass of the lamina.

Question # 5. [Marks: 4]

Find the center of the mass of the homogeneous solid that lies inside the cone $z=\sqrt{x^2+y^2}$ and the hemisphere $z=\sqrt{1-x^2-y^2}$.

Question # 6. [Marks: 5]

Use cylindrical coordinates to evaluate the integral

$$\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_0^{\sqrt{x^2+y^2}} \sqrt{x^2+y^2}\, dzdydx.$$

(1)

M-203

II Mid-term Examination (I Sem. 1433/1434)

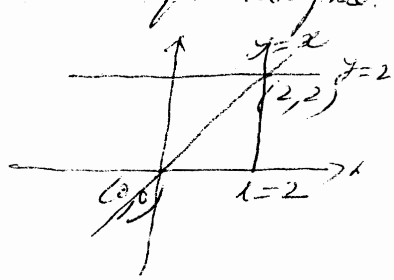
Max Time: 90 Minutes Max Marks: 25

Q#1) Evaluate the Integral $\int_0^2 \int_y^2 e^{x^2}\,dx\,dy$ [Marks: 4]

Soln. We shall do it by reversing the order of the integral.

Given $y \le x \le 2$

$0 \le y \le 2$



Reversing, we get $0 \le y \le x$

$0 \le x \le 2$

$$\therefore \int_0^2 \int_y^2 e^{x^2}\,dx\,dy = \int_0^2 \int_0^x e^{x^2}\,dy\,dx \quad (2)$$

$$= \int_0^2 e^{x^2}\left[y\right]_0^x dx$$

$$= \int_0^2 x e^{x^2}\,dx$$

Put $x^2 = t \Rightarrow$

$2x\,dx = dt$

$$= \frac{1}{2}\left[e^{x^2}\right]_0^2 \qquad \frac{1}{2}\int e^t\,dt$$

$$(2) = \frac{1}{2}\left[e^4 - 1\right] \qquad = \frac{1}{2}\left[e^t\right]$$

Q#2) Evaluate the integral using polar coordinates.

$$\iint_R \sqrt{4 - x^2 - y^2}\,dA \;; \qquad \text{[Marks: 4]}$$

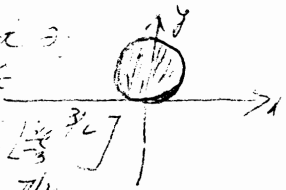
where R is the region bounded by the circle $x^2 + (y-1)^2 = 1$

Soln. $2\int_0^{\pi/2} \int_0^{2\sin\theta} \sqrt{4 - r^2}\; r\,dr\,d\theta$ (2)

Put $4 - r^2 = t \Rightarrow$

$-2r\,dr = dt$

$-\frac{1}{2}\int \sqrt{t}\,dt = -\frac{1}{2}\left[\frac{t^{3/2}}{3/2}\right]$



$$= -\frac{2}{3}\int_0^{\pi/2} \left[(4 - r^2)^{3/2}\right]_0^{2\sin\theta} d\theta$$

$$= -\frac{2}{3}\int_0^{\pi/2} \left[8 - 8\sin^3\theta\right] d\theta = \frac{8}{3}\pi - \frac{16}{3}\int_0^{\pi/2} \cos^3\theta\,d\theta$$

$$= \frac{8}{3}\pi - \frac{16}{3}\int_0^{\pi/2} (1 - \sin^2\theta)\cos\theta\,d\theta = \frac{8}{3}\pi - \frac{32}{9} \quad (2)$$

Put $\sin\theta = t \Rightarrow$

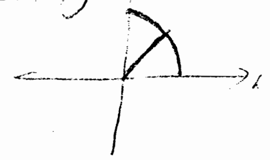
$\cos\theta\,d\theta = dt$

$\theta = 0,\ t = 0$

and $\theta = \pi/2,\ t = 1$

$\therefore -\frac{16}{3}\left[t - \frac{t^3}{3}\right]_0^1 = -\frac{32}{9}$

Q#3) Find the surface area of the portion of the surface $z = xy$ that is above the region in the first quadrant bounded by the lines $y = x$, $y = 0$, and the circle $x^2+y^2=9$ [Marks: 4]



Soln. We have $z = xy = f(x,y)$

$f_x = y$ and $f_y = x$

$\therefore$ Surface Area $S = \iint_R \sqrt{1+x^2+y^2}\, dA$

$$= \int_0^{\pi/4}\int_0^3 \sqrt{1+r^2}\; r\,dr\,d\theta$$

2 1

$$= \left(\frac{\pi}{4}\right)\left[\frac{1}{3}\left(10^{3/2}-1\right)\right]$$

$$= \frac{\pi}{12}\left(10^{3/2}-1\right)$$

$1+r^2 = t \Rightarrow$

$2r\,dr = dt$

$= \frac{1}{2}\int \sqrt{t}\,dt$

$= \frac{1}{2}\left[\frac{2}{3}t^{3/2}\right]$

$= \frac{1}{3}\left[(1+r^2)^{3/2}\right]$

$= \frac{1}{3}\left[10^{3/2}-1\right]$ 1

Q#4) The area mass density of the lamina in the shape of the region bounded by the graphs of $y^2 = x$ and $x+y=2$ is given by $\delta(x,y) = x$. Find the mass of the lamina. [Marks: 4]

Soln. Mass of the lamina $m = \iint_R \delta(x,y)\,dA$

$$= \int_{-2}^{1}\int_{y^2}^{2-y} x\,dx\,dy = \int_{-2}^{1}\left[\frac{x^2}{2}\right]_{y^2}^{2-y} dy$$

2

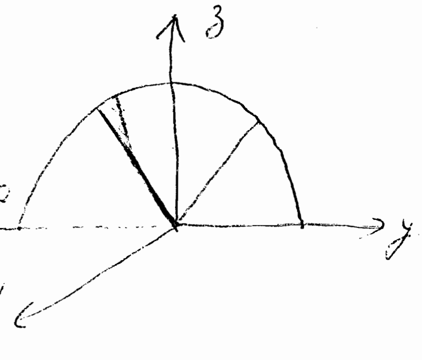
$$= \frac{1}{2}\int_{-2}^{1}\left[4-4y+y^2-y^4\right]dy$$

$$= \frac{1}{2}\left[4y-4\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^5}{5}\right]_{-2}^{1}$$

2

$$= \frac{68}{5}$$

Q # 5) Find the centre of the mass of the homogeneous solid that lies inside the cone $z=\sqrt{x^2+y^2}$ and the hemi-sphere $z=\sqrt{1-x^2-y^2}$ [Marks: 4]



Soln. we find first the volume

$$V=\iiint_Q dV=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta \quad (2)$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\varphi\, d\varphi\, d\theta$$

$$=\frac{1}{3}2\pi\left[-\cos\varphi\right]_0^{\pi/4}=\frac{2\pi}{3}\left[-\frac{1}{\sqrt2}+1\right]$$

$$=\frac{2\pi}{3}\left(\frac{\sqrt2-1}{\sqrt2}\right)$$

$$=\left(\frac{2-\sqrt2}{3}\right)\pi \quad (1)$$

Now, we find $\bar z=\dfrac{M_{xy}}{V}=\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 z\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

$$M_{xy}=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\cos\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta=\frac{\pi}{8}$$

∴ Now, the centre of the mass $(\bar x,\bar y,\bar z)$

$$=\left(0,0,\frac{3}{8(2-\sqrt2)}\right) \quad (1)$$

Q~~#~~6) Use cylindrical coordinates to evaluate the integral:

$$\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_0^{x^2+y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$$

[Marks: 5]

Soln.

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^{r^2} (r)\, r\, dz\, dr\, d\theta \quad 3$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^4\, dr\, d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2} 32\cos^5\theta\, d\theta$$

$$= \frac{64}{5}\int_0^{\pi/2} (1 - 2\sin^2\theta + \sin^4\theta)\cos\theta$$

Put $\sin\theta = t$, $\cos\theta\, d\theta = dt$. If $\theta = 0$, $t = 0$; if $\theta = \pi/2$, $t = 1$

$$= \frac{64}{5}\int_0^1 (1 - 2t^2 + t^4)\, dt$$

$$= \frac{64}{5} \times \frac{8}{15} = \frac{512}{75} \quad 2$$