

Department of Mathematics
College of Sciences
King Saud University, Riyadh.

M-203 (Differential and Integral Calculus)

1st MidTerm Examination (2nd semester 1446) (2024/2025),

Time: 90 Minutes

Max. Marks: 25.

Note: All questions carry equal marks.

Q1. Determine whether the sequence $\left\{ \frac{3n^2 + n \sin(n) + 1}{5n \cos(n) + 2n^2 + 1} \right\}$ is convergent or divergent, and if it is convergent find its limit.

Q2. Find the sum of the series:

$$\sum_{n=1}^{\infty} \frac{3}{(3n-1)(3n+2)}.$$

Q3. Determine whether the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$ is absolutely convergent, conditionally convergent, or divergent.

Q4. Find the interval of convergence and the radius of

convergence of the power series: $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n2^n}.$

Q5. Find the power series representation of the function

$f(x) = \tan^{-1}(3x^2)$ and use the first ^{two} ~~three~~ nonzero terms to

approximate the integral $\int_0^1 \tan^{-1}(3x^2) dx.$

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Solutions to the Questions

Q #1) Determine whether the sequence $\left\{ \frac{3n^2 + n \sin(n) + 1}{5n \cos(n) + 2n^4 + 1} \right\}$ is convergent or divergent, and if it is convergent find its limit. [Marks: 5]

Soln. We have $a_n = \frac{3n^2 + n \sin(n) + 1}{5n \cos(n) + 2n^4 + 1}$

Dividing the numerator and denominator by n^4 , we get

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{3 + \frac{\sin(n)}{n} + \frac{1}{n^2}}{2 + 5 \frac{\cos(n)}{n} + \frac{1}{n^2}} = \frac{3}{2} \quad (3)$$

Since $0 \leq \left| \frac{\sin n}{n} \right| \leq \frac{1}{n}$ and $0 \leq \left| \frac{\cos n}{n} \right| \leq \frac{1}{n}$ and $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = \lim_{n \rightarrow \infty} \frac{\cos n}{n} = 0$. Hence $\lim_{n \rightarrow \infty} a_n = \frac{3}{2}$; Convg. (1) (1)

Q #2) Find the sum of the series: $\sum_{n=1}^{\infty} \frac{3}{(3n-1)(3n+2)}$ [Marks: 5]

Soln. we have $a_n = \frac{3}{(3n-1)(3n+2)}$

$$\text{Now } \frac{3}{(3n-1)(3n+2)} = \frac{A}{3n-1} + \frac{B}{3n+2} = \frac{A(3n+2) + B(3n-1)}{(3n-1)(3n+2)}$$

Put $n = \frac{1}{3}$. we have $3 = 3A \therefore A = 1$ (2)

Put $n = -\frac{2}{3}$. we have $3 = 3B \therefore B = -1$

Hence, we have $\frac{3}{(3n-1)(3n+2)} = \frac{1}{3n-1} - \frac{1}{3n+2}$

$$\therefore \sum_{n=1}^{\infty} \frac{3}{(3n-1)(3n+2)} = \sum_{n=1}^{\infty} \left[\frac{1}{3n-1} - \frac{1}{3n+2} \right]$$

$$= \left[\left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{8} \right) + \left(\frac{1}{8} - \frac{1}{11} \right) + \dots + \left(\frac{1}{3n-1} - \frac{1}{3n+2} \right) \right]$$

$$\therefore \sum_{n=1}^{\infty} \frac{3}{(3n-1)(3n+2)} = \lim_{n \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{3n+2} \right] = \frac{1}{2} \quad \textcircled{1}$$

Q#3) Determine whether the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$ is absolutely convergent, conditionally convergent or div. [Marks: 5]

Soln. $\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{\ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

and $\frac{1}{\ln(n)} > \frac{1}{n} \forall n$ and $\sum \frac{1}{n}$ is div. (2)

Hence $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ is divergent

But $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$ is convergent by AST. (2)

Hence the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$ is conditionally conv. (1)

Q#4) Find the interval of convergence and the radius of conv. for the power series $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n \cdot 2^n}$ [Marks: 5]

Soln. we apply absolute Ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{(2n+1)^{n+1}}{(n+1) 2^{n+1}} \times \frac{n \cdot 2^n}{(2n+1)^n} \right| = \frac{1}{2} \left| \frac{2n+1}{n+1} \right|$$

For abs. conv. $\frac{1}{2} \left| \frac{2n+1}{n+1} \right| < 1 \Rightarrow -2 < 2n+1 < 2$

$\Rightarrow -3 < 2n < 1 \Rightarrow -\frac{3}{2} < n < \frac{1}{2}$ (2)

At $x = -\frac{3}{2}$, we have $\sum_{n=1}^{\infty} \frac{\left(2\left(-\frac{3}{2}\right)+1\right)^n}{n 2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ which (1)

is $\log$ by AST

At $x = \frac{1}{2}$, we have $\sum_{n=1}^{\infty} \frac{1}{n}$ which is $\log$. (1)

Hence the Interval of $\log$: $\left[-\frac{3}{2}, \frac{1}{2}\right)$.

Radius of convergence: $r = 1$. (1)

Q.45) Find the power series representation of the function $f(x) = \tan^{-1}(3x^2)$ and use the first three non-zero terms to approximate the integral $\int_0^1 \tan^{-1}(3x^2) dx$. [Marks: 5]

Soln. We know $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}$ (2)

Replacing x by $3x^2$, we get

$$\begin{aligned} \tan^{-1}(3x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (3x^2)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} 3^{2n+1} x^{4n+2} \quad (1) \end{aligned}$$

$$\therefore \tan^{-1}(3x^2) = \frac{3x^2}{1} - \frac{3^3 x^6}{3} + \frac{3^5 x^{10}}{5} - \dots$$

$$\begin{aligned} \text{Now } \int_0^1 \tan^{-1}(3x^2) dx &= \int_0^1 \left(3x^2 - 9x^6 + \frac{243}{5} x^{10} - \dots \right) dx \\ &= \left[\frac{3x^3}{3} - \frac{9 \cdot \frac{x^7}{7}}{7} + \frac{243}{5} \frac{x^7}{7} + \frac{243}{5} \frac{x^{11}}{11} - \dots \right]_0^1 \\ &= 1 - \frac{9}{7} + \frac{243}{55} = 1 - 1.2857 + 4.4182 \\ &\quad (1) \qquad \qquad \qquad \approx 4.1325 (1) \end{aligned}$$