Department of Mathematics
College of Sciences
King Saud University, Riyadh.

M-203 (Differential and Integral Calculus)
1^{st} MidTerm Examination (1^{st} semester 1448) (2026/2027),
Time: **90 Minutes** Max. Marks: 25.
Note: **All questions carry equal marks.**

Q1. Determine whether the sequence $\left\{n\left[\frac{2}{n+1}+\sin\left(\frac{1}{n}\right)\right]\right\}$ is convergent or divergent, and if it is convergent find its limit.

Q2. Show that the following series is convergent

$$\sum_{n=2}^{\infty}\left[\left(\frac{-2}{5}\right)^n+\frac{\ln(n)}{n^2\sqrt{n}}\right].$$

Q3. Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{(\sqrt{n}+1)(\sqrt{n}+2)}$ is absolutely convergent, conditionally convergent, or divergent.

Q4. Find the radius of convergence and the interval of convergence of the power series: $\sum_{n=0}^{\infty}\frac{n(x-3)^n}{4^{n+1}}$.

Q5. Find the Maclaurin series for the function $f(x)=\sin(x^2)$ and use the first three nonzero terms to approximate the integral

$$\int_0^{0.5}\frac{\sin(x^2)}{x}dx.$$

Solutions of 1^{st} MidTerm Exam. (1^{st} sem. 1448 (2026/2027)

Q1. Find the limit $\lim_{n\to\infty} n\left[\frac{2}{n+1}+\sin\left(\frac{1}{n}\right)\right]$.

Solution. Split the expression into two parts:

$$n\left[\frac{2}{n+1}+\sin\left(\frac{1}{n}\right)\right]=\frac{2n}{n+1}+n\sin\left(\frac{1}{n}\right). \qquad \text{0.5 pt}$$

For the first part,

$$\frac{2n}{n+1}=\frac{2}{1+\frac{1}{n}}\xrightarrow[n\to\infty]{}2. \qquad \text{2 pts}$$

For the second part, put $t=\frac{1}{n}\to 0^+$; using the standard limit $\frac{\sin t}{t}\to 1$,

$$n\sin\left(\frac{1}{n}\right)=\frac{\sin t}{t}\xrightarrow[t\to 0]{}1. \qquad \text{2pts}$$

Adding the two limits,

$$\boxed{\lim_{n\to\infty} n\left[\frac{2}{n+1}+\sin\left(\frac{1}{n}\right)\right]=2+1=3.} \qquad \text{0.5 pt}$$

Q2. Show that the series $\sum_{n=2}^{\infty}\left[\left(\frac{-2}{5}\right)^n+\frac{\ln n}{n^2\sqrt{n}}\right]$ converges.

Solution. We show that each of the two series $\sum_{n=2}^{\infty}\left(\frac{-2}{5}\right)^n$, $\sum_{n=2}^{\infty}\frac{\ln n}{n^{5/2}}$ converges separately.

First series (geometric). It is geometric with ratio $r=-\frac{2}{5}$, and $|r|=\frac{2}{5}<1$, hence it converges. 2pts

Second series (positive terms). Here $\frac{\ln n}{n^2\sqrt{n}}=\frac{\ln n}{n^{5/2}}>0$, $\forall n\geq 2$. Since $\ln n\leq n$, $\forall n\geq 2$ we have

$$\frac{\ln n}{n^{5/2}}\leq\frac{n}{n^{5/2}}=\frac{1}{n^{\frac{3}{2}}}.$$

The series $\sum_{n=2}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is a convergent p-series ($p=\frac{3}{2}>1$), so by the basic comparison test (BCT) $\sum_{n=2}^{\infty}\frac{\ln n}{n^{5/2}}$ converges. 3pts

Conclusion. Both series converge, hence their sum converges: $\sum_{n=2}^{\infty}\left[\left(\frac{-2}{5}\right)^n+\frac{\ln n}{n^2\sqrt{n}}\right]$ converges.

Q3. Determine whether $\sum_{n=1}^{\infty}\frac{(-1)^n}{(\sqrt{n}+1)(\sqrt{n}+2)}$ is absolutely convergent, conditionally convergent, or divergent.

Solution. Write $a_n = \dfrac{(-1)^n}{(\sqrt{n}+1)(\sqrt{n}+2)}$ and $b_n = |a_n| = \dfrac{1}{(\sqrt{n}+1)(\sqrt{n}+2)} > 0.$

Absolute convergence? Compare b_n with $\dfrac{1}{n}$ using the limit comparison test (LCT):

$$\lim_{n\to\infty} \frac{b_n}{1/n} = \lim_{n\to\infty} \frac{n}{(\sqrt{n}+1)(\sqrt{n}+2)} = \lim_{n\to\infty} \frac{n}{n+3\sqrt{n}+2} = 1. \quad \text{2pts}$$

The limit is finite and nonzero, and $\sum 1/n$ (the harmonic series) diverges, so $\sum b_n$ *diverges.* Hence the series is **not** absolutely convergent.

Conditional convergence? Apply the alternating series test (AST) to $\sum(-1)^n b_n$.

- $b_n \to 0$, since $(\sqrt{n}+1)(\sqrt{n}+2) \to \infty$.
- b_n is decreasing: both factors $\sqrt{n}+1$ and $\sqrt{n}+2$ increase with n, so their product increases and its reciprocal b_n decreases.

Both conditions hold, so by the alternating series test (AST) the series converges. 2pts
Conclusion. The series converges but not absolutely, so it is **conditionally convergent.** 1pt

Q4. Find the radius and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n\,(x-3)^n}{4^{n+1}}$.

Solution. Let $u_n = \dfrac{n}{4^{n+1}}(x-3)^n$. Apply the absolute ratio test:

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{(n+1)\,|x-3|^{n+1}/4^{n+2}}{n\,|x-3|^n/4^{n+1}} = \frac{n+1}{n}\cdot\frac{|x-3|}{4} \xrightarrow[n\to\infty]{} \frac{|x-3|}{4}.$$

The series converges when this limit is < 1, i.e.

$$\frac{|x-3|}{4} < 1 \iff |x-3| < 4. \quad \text{2pts}$$

1pt Hence the **radius of convergence is** $R = 4$, and the series converges at least on the open interval $3-4 < x < 3+4$, that is $(-1, 7)$.

- At $x = 7$: we have $\displaystyle\sum_{n=0}^{\infty} \frac{n\,(x-3)^n}{4^{n+1}} = \sum_{n=0}^{\infty} \frac{n\cdot 4^n}{4^{n+1}} = \sum_{n=0}^{\infty} \frac{n}{4}$. Since $\dfrac{n}{4} \to \infty \neq 0$, by the nth term the series diverges. 1pt
- At $x = -1$: we have $\displaystyle\sum_{n=0}^{\infty} \frac{n\,(x-3)^n}{4^{n+1}} = \sum_{n=0}^{\infty} \frac{n\cdot(-4)^n}{4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{4}$. Again the terms do not tend to 0, so the series diverges. 1pt

Consequently, the radius of convergence $R = 4$, and the interval of convergence is $(-1, 7)$.

Q5. Find the Maclaurin series of $f(x) = \sin(x^2)$ and use its first three nonzero terms to approximate $\displaystyle\int_0^{0.5} \frac{\sin(x^2)}{x}\,dx$.

Solution. We know that the Maclaurin series of the function $\sin(u)$,

$$\sin(u) = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{(2k+1)!} = u - \frac{u^3}{6} + \frac{u^5}{120} - \cdots. \quad \text{1pt}$$

Substitute $u = x^2$:

$$\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots. \qquad \text{1pt}$$

Dividing by x and using the first three nonzero terms,

$$\frac{\sin(x^2)}{x} \approx x - \frac{x^5}{6} + \frac{x^9}{120}.$$

Integrating term by term,

$$\int_0^{0.5} \frac{\sin(x^2)}{x}\,dx \approx \int_0^{0.5} \left(x - \frac{x^5}{6} + \frac{x^9}{120}\right) dx = \left[\frac{x^2}{2} - \frac{x^6}{36} + \frac{x^{10}}{1200}\right]_0^{0.5}. \qquad \text{2pts}$$

With $x = 0.5 = \frac{1}{2}$:

$$\frac{(1/2)^2}{2} = \frac{1}{8} = 0.125, \qquad \frac{(1/2)^6}{36} = \frac{1}{2304} \approx 0.000434, \qquad \frac{(1/2)^{10}}{1200} = \frac{1}{1\,228\,800} \approx 8.1 \times 10^{-7}.$$

Therefore

$$\int_0^{0.5} \frac{\sin(x^2)}{x}\,dx \approx 0.125 - 0.000434 + 0.0000008 \approx \boxed{0.12457}. \qquad \text{1pt}$$