Department of Mathematics, College of Sciences
King Saud University, Riyadh.

M-203 (Differential and Integral Calculus)
1^{st} MidTerm Exam. (1^{st} sem. 1447) (2025/2026),
Time: **90 Minutes** Max. Marks: 25.
Note: **All questions carry equal marks.**

Q1. Determine whether the sequence $\left\{\left(\frac{n+2}{n}\right)^n\right\}_{n=1}^{\infty}$ converges or diverges, and if it converges find its limit.

Q2. Find the sum of the series:

$$\sum_{n=0}^{\infty}\left[(-1)^n\left(\frac{2^{2n+1}}{5^n}\right)\right]+\sum_{n=3}^{\infty}\left[\frac{1}{n(4n+8)}\right].$$

Q3. Determine whether the following series is absolutely convergent, conditionally convergent, or divergent: $\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\sqrt{\ln(n)}}$.

Q4. Find the interval of convergence and the radius of convergence of the power series: $\sum_{n=1}^{\infty}\frac{(x+10)^n}{n2^n}$.

Q5. Find the power series representation of the function $f(x)=\frac{-x}{x^3-8}$ and then use the first three non-zero terms to approximate the integral $\int_0^1\frac{-x^{\frac{1}{3}}}{x-8}dx$.

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Solutions to the Questions.

Q#1) Determine whether the sequence $\left\{\left(\frac{n+2}{n}\right)^n\right\}_{n=1}^{\infty}$ converges or diverges, and if it converges, find its limit. [Marks: 5]

Sol. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$: 1^∞ form.

Let $\lim y = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$: 1^∞ 1

$\lim \ln y = \lim_{n\to\infty} n \ln\left(1+\frac{2}{n}\right)$: $\infty\cdot 0$ form. 1

$= \lim_{n\to\infty} \dfrac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}$: $\frac{0}{0}$ form 1

Apply L'Hopital rule: $\lim_{n\to\infty} \dfrac{\frac{1}{1+\frac{2}{n}}\left(-\frac{2}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} = 2.$ 1

$\therefore$ $\lim y = \lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n = e^2$, conv. 1

Q#2). Find the sum of the Series: $\underbrace{\sum_{n=0}^{\infty}\left[(-1)^n \frac{2^{n+1}}{5^n}\right]}_{S_1} + \underbrace{\sum_{n=3}^{\infty}\left[\frac{1}{n(4n+\dots}\right.}_{S_2}$ [Marks: 5]

$S_1 = \sum_{n=0}^{\infty} (-1)^n \frac{2^{n+1}}{5^n} = 2\left[\frac{1}{1+\frac{4}{5}}\right] = \frac{10}{9}$ 2

$$S_2 = \sum_{n=3}^{\infty} \left[\frac{1}{n(4n+8)} \right]$$

$$\sum \frac{1}{n(4n+8)} = \sum \frac{1}{4n(n+2)} \; ; \; \frac{1}{n(+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$S_2 = \frac{1}{8} \sum_{n=3}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{8} \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{8} \left[\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+2} \right]$$

$$= \lim_{n\to\infty} \frac{1}{8} \left(\frac{1}{3} + \frac{1}{4} \right) + \cdots \underbrace{\left(\frac{1}{n} - \frac{1}{n+2} \right)}_{=0}$$

$$= \frac{1}{8} \left(\frac{7}{12} \right) = \frac{7}{96} \quad 2$$

$$\therefore S_1 + S_2 = \frac{10}{9} + \frac{7}{96} = \frac{341}{288} \quad 1$$

Q#3) Determine whether the following series is abs. cong., conditionally cong. or divergent: $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\sqrt{\ln(n)}}$. [Marks: 5]

Sol. $a_n = \frac{1}{n\sqrt{\ln(n)}} > 0$ and $\frac{a_{n+1}}{a_n} = \frac{n\sqrt{\ln(n)}}{(n+1)\sqrt{\ln(n+1)}} < 1$

and $\lim_{n\to\infty} a_n = 0$. Hence by AST, $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\sqrt{\ln(n)}}$ is cong. 2

But $\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{n\sqrt{\ln(n)}} \right| = \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

By Integral test: $\lim_{t\to\infty} \int_2^t \frac{1}{x\ln(x)}\,dx$

Also, from the fact that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^k}$ is cong. if $k > 1$ and divg. if $k < 1$. <u>elicc</u>

Put $\ln(x) = u$ & $\frac{1}{x}dx = du$

$\int \frac{1}{\sqrt{u}}\,du = 2(\ln x)^{\frac{1}{2}}$

Hence $\lim_{x\to t} \int_2^t \frac{1}{x\ln(x)}\,dx$ is divg. 2

Hence it is not absolutely cong. Hence the series Conditionally cong. 1

Q#4) Find the Interval of convergence and the radius of convergence of the power Series: $\sum_{n=1}^{\infty} \frac{(x+10)^n}{n\,2^n}$. [Marks:5]

Soln. Applying Absolute Ratio test, we have

$$\lim_{n\to\infty} \left| \frac{(x+10)^{n+1}}{(n+1)2^{n+1}} \times \frac{n\,2^n}{(x+10)^n} \right|$$

$$= \frac{1}{2} |x+10|$$

For abs. Cong. $\frac{1}{2}|x+10| < 1 \iff |x+10| < 2$

$\iff -2 < x+10 < 2 \iff -12 < x < -8$ (2)

At $x = -12$, we have $\sum_{n=1}^{\infty} \frac{(-2)^n}{n\,2^n} = \sum (-1)^n \frac{1}{n}$ which is Convergent by AST. (1)

At $x = -8$, we have $\sum_{n=1}^{\infty} \frac{2^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent harmonic Series. (1)

Hence Interval of Cong: $[-12, -8)$

Radius of cong: $\frac{-8-(-12)}{2} = \underline{2}$ (1)

Q#5) Find the power Series representation of the function $f(x) = \frac{-x}{x^3-8}$ and then use the first three non-zero terms to approximate $\int_0^1 \frac{-x^{\frac{1}{3}}}{x-8} dx$.

[Marks: 5]

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Soln. $f(x) = \frac{-x}{x^3-8} = \frac{x}{8-x^3} = \frac{x}{8}\frac{1}{1-\frac{x^3}{8}} = \frac{x}{8}\sum_{n=0}^{\infty}\left(\frac{x^3}{8}\right)^n$

$= \frac{x}{8}\sum_{n=0}^{\infty}\frac{x^{3n}}{8^n} = \sum_{n=0}^{\infty}\frac{x^{3n+1}}{8^{n+1}}$

$\approx \frac{x}{8} + \frac{x^4}{8^2} + \frac{x^7}{8^3} + \cdots$ (2)

So $\frac{-x^{\frac{1}{3}}}{(x-8)} \approx + \frac{x^{-\frac{1}{3}}}{x-8} \approx \frac{x^{\frac{1}{3}}}{8} + \frac{x^{\frac{4}{3}}}{64} + \frac{x^{7/3}}{512}$ (1)

$\therefore \int_0^1 -\frac{x^{\frac{1}{3}}}{x-8}\,dx = \int_0^1\left(\frac{x^{\frac{1}{3}}}{8} + \frac{x^{\frac{4}{3}}}{64} + \frac{x^{7/3}}{512}\right)dx$

$= \left[\frac{1}{8}\times\frac{3}{4}x^{4/3} + \frac{1}{64}\times\frac{3}{7}x^{7/3} + \frac{3}{10}\times\frac{1}{512}x^{\frac{10}{3}}\right]_0^1$

$\approx \frac{3}{32} + \frac{3}{448} + \frac{3}{5120} \approx \underline{0.10103}$ (2)