King Saud University, College of Science
Department of Mathematics
M - 203, First Mid-Term Examination
First Semester, 1446
Time: 90 Minutes Max. Marks: 25

Note: All questions carry equal marks.

Q#1) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{4}{7^{n-1}} + \sum_{n=4}^{\infty} \frac{12}{(2n-6)(2n-4)}.$$

Q#2) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{|\cos n|}{\sqrt{3+n^2+2n^3}}.$$

is convergent or divergent.

Q#3) Find the interval of convergence and radius of convergent for the power series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}(x-1)^n.$$

Q#4) Find the power representation for the function $f(x) = \frac{1}{1+x^2}$, if $-1 < x < 1$ and use it to find power series representation of $\tan^{-1} x$, if $-1 \leq x \leq 1$.

Q#5) Find a Maclaurin series for the function $f(x) = e^x$ and approximate the integral

$$\int_0^{0.1} x^2 e^{-x^2}\, dx$$

by using the first three non-zero terms.

King Saud University, College of Science (1)

Department of Mathematics

M-203 First Mid-term Examination

Time: 90 Mts. First semester 1446 Max. Marks: 25

Q#1) Find the sum of the series: $\sum_{n=1}^{\infty} \frac{4}{7^{n-1}} + \sum_{n=4}^{\infty} \frac{12}{(2n-6)(2n-4)}$ (5 Marks)

Soln. We find $S_1 = \sum_{n=1}^{\infty} \frac{4}{7^{n-1}} = 4\sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^{n-1}$

$$= 4 \times \frac{1}{1-\frac{1}{7}} = \cancel{4}^{2} \times \frac{7}{\cancel{6}_3} = \frac{14}{3}$$ (2)

$$S_2 = \sum_{n=4}^{\infty} \frac{12}{(2n-6)(2n-4)} = \sum_{n=4}^{\infty} \frac{\cancel{12}^{3}}{\cancel{2\times 2}(n-3)(n-2)} = \frac{3}{(n-3)(n-2)}$$

Now, $\frac{3}{(n-3)(n-2)} = \frac{A}{n-3} + \frac{B}{n-2}$

$$= \frac{A(n-2) + B(n-3)}{(n-3)(n-2)}$$

We get $A = 3$ and $B = -3$

$\therefore \frac{3}{(n-3)(n-2)} = 3\left[\frac{1}{n-3} - \frac{1}{n-2}\right]$

$$\therefore 3\sum_{4}^{n} \left[\left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cdots + \left(\cancel{\tfrac{1}{n-3}} - \frac{1}{n-2}\right)\right]$$

$$= 3 \sum_{4}^{\infty} \left[1 - \frac{1}{n-2}\right]$$

$$= 3 \lim_{n\to\infty} (1) = 3$$ (2)

Hence sum $S = \frac{14}{3} + 3 = \frac{14+9}{3} = \frac{23}{3}$ (1)

Q#2) Determine whether the Series $\sum_{n=1}^{\infty} \frac{|\cos n|}{\sqrt{3+n^2+2n^3}}$ is Convergent or divergent. [Marks: 5]

Soln.
$$\frac{|\cos n|}{\sqrt{3+n^2+2n^3}} \le \frac{1}{\sqrt{3+n^2+2n^3}} \le \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}} \quad \forall n$$ 3

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent by p-series test 1

Hence by Basic Comparison test $\sum_{n=1}^{\infty} \frac{|\cos n|}{\sqrt{3+n^2+2n^3}}$ is convergent. 1

Q#3) Find the interval of convergence and radius of convergence of the Power Series $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}(x-1)^n$ [Marks: 5]

Soln. By Absolute Ratio test, we have

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)(\ln(n+1))^2}(x-1)^{n+1} \times n(\ln(n))^2 \cdot \frac{1}{(x-1)^n} \right|$$

$$= |x-1| \iff -1 < x-1 < 1$$
$$\iff 0 < x < 2$$ 2

At $x=0$, we have $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}(-1)^n$ which is cong by AST 1

At $x=2$, we have $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ which is cong by 1 Integral test. Hence Interval of cong: $[0, 2]$.

Radius of cong: $\frac{2-0}{2} = 1$ 1

CamScanner الممسوحة ضوئيا بـ

Q#4) Find the power series representation for the function $f(x)=\frac{1}{1+x^2}$ if $-1<x<1$ and use it to find power series representation of $\tan^{-1}x$ if $-1\le x\le 1$. [Marks: 5]

Soln. We know, if $|x|<1$

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots \quad (1)$$

Now replacing x by x^2, we get

$$\frac{1}{1+x^2}=1-x^2+(x^2)^2-(x^2)^3+\cdots+(-1)^n(x^2)^n+\cdots$$

$$=1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots \quad (2)$$

We see that

$$\tan^{-1}x=\int_0^x \frac{1}{1+t^2}\,dt \quad (1)$$

$$=\int_0^x\left[1-t^2+t^4-t^6+\cdots+(-1)^n t^{2n}-\cdots\right]dt$$

$$=\left[t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\cdots+(-1)^n\frac{t^{2n+1}}{2n+1}\cdots\right]_0^x$$

$$=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots \quad (1)$$

$-1\le x\le 1$.

Q#5). Find a Maclaurin Series for the function $f(x)=e^x$ and approximate the integral $\int_0^{0.1} x^2 e^{-x^2}\,dx$ by using the first three non-zero terms. [Marks: 5]

Soln: We have $f(x)=e^x \Rightarrow f(0)=e^0=1$.

$f'(x)=e^x \Rightarrow f'(0)=1$.

$f''(x)=e^x \Rightarrow f''(0)=1$

$\vdots$

Hence $f(x)=e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots \quad (2)$

$$\therefore \int_0^1 x^2 e^{-x^2}\,dx = \int_0^{0.1} x^2\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right] dx$$

$$= \int_0^{0.1}\left(x^2 - x^4 + \frac{x^6}{2!} - \cdots\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7(2!)}\right]_0^{0.1} \quad 2$$

$$= \frac{(0.1)^3}{3} - \frac{(0.1)^5}{5} + \frac{(0.1)^7}{7(2!)}$$

$$= 0.000333 - 0.000002 + \frac{0.0000001}{7\times 2}$$

1