FINAL EXAMINATION
SEMESTER I: 1433-1434
Department of Mathematics
King Saud University
MATH: 203 Time: 3 Hours Full Marks: 40

Question # 1. Marks: 2+3+4=9

(a) Determine whether the sequence $\{\sqrt{n^2+n}-n\}$ converges or diverges and if converges, find its limit.

(b) Determine whether the series $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n^2}$ converges or diverges.

(c) Find the interval of convergence and radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x+2)^n$.

Question # 2. Marks: 3+3+3=9

(a) Find the Maclaurin series of $\sin x$ and use it to approximate the integral $\int_{-1}^{1} \frac{\sin x}{x} dx$ to four decimal places.

(b) Evaluate the integral $\int_0^1 \int_x^1 \sin(y^2) dy dx$.

(c) Evaluate the integral $\int\int_R (3x+4y^2) dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Question # 3. Marks: 3+3+3=9

(a) A solid is bounded by $x=0$, $y=0$, $z=0$ and the plane $x+y+z=1$ and has the density $\delta(x,y,z)=z$ at any point $P(x,y,z)$. Find the mass of the solid.

(b) By changing to spherical coordinates, evaluate the integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dz dy dx.$$

(c) Evaluate the line integral $\int_C x dy - y dx$, where C is the portion of the circle $x^2+y^2=1$ from the point $(0,-1)$ to the point $(0,1)$.

Question # 4. Marks: 4+4+5=13

(a) Show that the line integral $\int_C x dy + y dx$ is independent of path and evaluate the integral $\int_{(-1,1)}^{(1,1)} x dy + y dx$.

(b) If the vector force $\vec{F}(x,y,z) = x\vec{i} - y\vec{j} + z\vec{k}$ and S is the surface of the sphere $x^2+y^2+z^2=1$, using divergence theorem find the flux of $\vec{F}$ through the surface S.

(c) If $\vec{F} = x\vec{i} - y\vec{j} + z\vec{k}$ and S is the surface of the hemisphere $z=\sqrt{9-x^2-y^2}$, verify the Stokes' theorem.

Solutions:

Fam (1

Q1:

(a) $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)=\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}$

$=\lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\lim_{n\to\infty}\frac{n}{\sqrt{n^2(1+\frac{1}{n})}+n}$

$=\lim_{n\to\infty}\frac{n}{n\sqrt{1+\frac{1}{n}}+n}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{2}$ (Conv)

(b) $\sum\frac{\sin(\frac{1}{n})}{n^2}\leq\sum\frac{1}{n^2}\longrightarrow$ P-series $P=2>1$ $\Rightarrow$ Conv

By B.C.T $\Rightarrow$ $\sum\frac{\sin(\frac{1}{n})}{n^2}$ Conv

(c) $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^2}(x+2)^{n+1}\cdot\frac{n^2}{2^n(x+2)^n}\right|$

$=\lim_{n\to\infty}\frac{\cancel{2^n}\cdot 2}{n^2+2n+1}\cdot\frac{n^2}{\cancel{2^n}}\left|\frac{\cancel{(x+2)^n}(x+2)}{\cancel{(x+2)^n}}\right|$

$=\lim_{n\to\infty}\frac{2n^2}{n^2+2n+1}|x+2|=2|x+2|<1$

$\Rightarrow |x+2|<\frac{1}{2}\Rightarrow -\frac{1}{2}<x+2<\frac{1}{2}\Rightarrow -\frac{5}{2}<x<-\frac{3}{2}$

Check at $x=-\frac{3}{2}$: $\sum\frac{2^n}{n^2}\cdot\frac{1}{2^n}=\sum\frac{1}{n^2}$ P-series $P=2>1$ Conv

Check at $x=-\frac{5}{2}$: $\sum\frac{2^n}{n^2}\left(-\frac{1}{2}\right)^n=\sum\frac{1}{n^2}(-1)^n$ A.S

$a_n=\frac{1}{n^2}>0$, $\lim_{n\to\infty}\frac{1}{n^2}=0$, $a_n'=-\frac{2}{n^3}<0\Rightarrow a_n\downarrow$

By A.S.T $\Rightarrow\sum\frac{(-1)^n}{n^2}$ Conv

$\therefore$ Interval of convergence $=\left[-\frac{5}{2},-\frac{3}{2}\right]$

radius of convergence $=\frac{-\frac{3}{2}+\frac{5}{2}}{2}=\frac{1}{2}$

Q2:

(a) $f(x) = \sin x \rightarrow f(0) = 0$

$f'(x) = \cos x \rightarrow f'(0) = 1$

$f''(x) = -\sin x \rightarrow f''(0) = 0$

$f'''(x) = -\cos x \rightarrow f'''(0) = -1$

Maclaurin series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)(x-0)^n}{n!}$

$= \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\Rightarrow \int_{-1}^{1} \frac{\sin x}{x} dx = \int_{-1}^{1} \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots\right) dx$

$= 2\int_0^1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots\right) dx = 2\left[x - \frac{x^3}{18} + \frac{x^5}{600} - \cdots\right]_0^1$

$= 2\left[1 - \frac{1}{18} + \frac{1}{600} - \cdots\right] = 2 - \frac{1}{9} + \frac{1}{300} - \frac{1}{17640} + \cdots$

$= 2 - 0.\overline{11} + 0.00\overline{3} + 0.0000567 + \cdots$

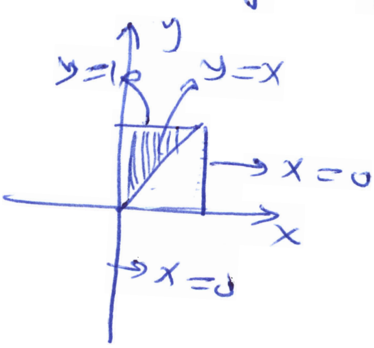
$\approx 2 - 0.\overline{11} + 0.00\overline{3} = 1.8933$

(b)

We have to reverse the order of integration:

$0 \leq x \leq 1, \quad x \leq y \leq 1$

$\Rightarrow 0 \leq y \leq 1, \quad 0 \leq x \leq y$

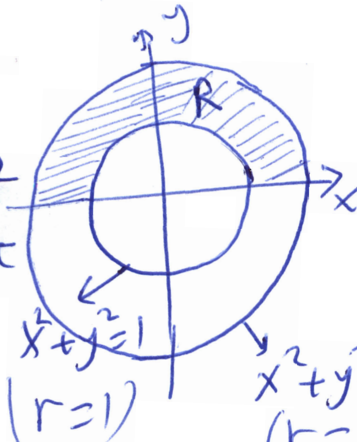


$\Rightarrow I = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy$

$= \int_0^1 x \sin(y^2)\Big]_0^y dy = \int_0^1 y \sin(y^2)\, dy$

$= -\frac{1}{2} \cos(y^3)\Big]_0^1 = \frac{1}{2} - \frac{1}{2}\cos(1)$

(c)



$1 \le r \le 2$

$0 \le \theta \le \pi$

$$I = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{\pi} r^3\cos\theta + r^4\sin^2\theta \Big]_1^2 \,d\theta$$

$$= \int_0^{\pi} (8\cos\theta + 16\sin^2\theta - \cos\theta - \sin^2\theta)\,d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\,d\theta$$

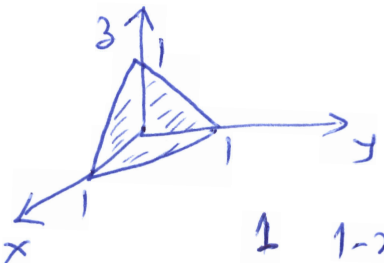
$$= 7\sin\theta\Big]_0^{\pi} + \frac{15}{2}\int_0^{\pi} [1 - \cos(2\theta)]\,d\theta$$

$$= 0 + \frac{15}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi}$$

$$= \frac{15}{2}\pi$$

Q#3:

(a)



$$m = \iiint_Q \delta\, dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}z^2\Big]_0^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)^2\,dy\,dx = \int_0^1 -\frac{1}{6}(1-x-y)^3\Big]_0^{1-x} dx$$

$$= -\frac{1}{6}\int_0^1 [0 - (1-x)^3]\,dx = -\frac{1}{24}(1-x)^4\Big]_0^1 = 1/24$$

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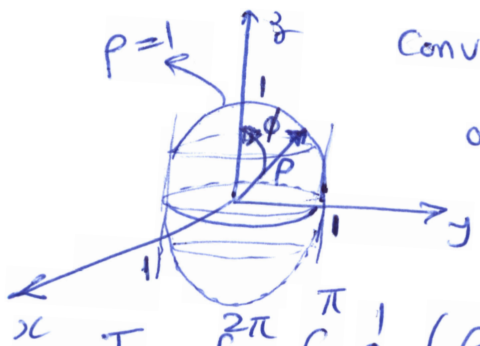
(b) $\left\{-1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ -\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}\right\} \in \varphi$



Convert to spherical coordinates:

$0 \le \rho \le 1,\quad 0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{(\rho^2)^{\frac{3}{2}}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 e^{\rho^3} \sin\phi \, d\rho \, d\phi \, d\theta$$

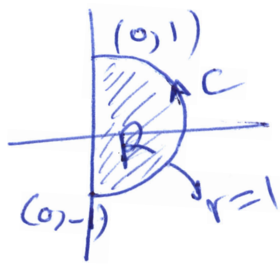
$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3} e^{\rho^3}\Big]_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{1}{3}e - \frac{1}{3}\right) \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{3} - \frac{1}{3}e\right) \cos\phi \Big]_0^{\pi} d\theta = \int_0^{2\pi} \left(\frac{2}{3}e - \frac{2}{3}\right) d\theta$$

$$= \left(\frac{2}{3}e - \frac{2}{3}\right)\theta \Big]_0^{2\pi} = \frac{4\pi}{3}e - \frac{4\pi}{3} = \frac{4\pi}{3}(e-1)$$

(c) Green's Theorem: $\oint_C M\,dx + N\,dy = \iint_{R_{xy}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$



$M = -y,\ N = x$

$\frac{\partial M}{\partial y} = -1,\quad \frac{\partial N}{\partial x} = 1$

$$\int_C x\,dy - y\,dx = \iint_R 2\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 2\,r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} r^2\Big]_0^1 d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi$$

Q # 4:

(a) $M = y$, $N = x$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ the line integral is independent of path.

$$f_x = M = y \Rightarrow f(x,y) = \int y\,dx = xy + c_1(y)$$

$$f_y = N = x \Rightarrow$$

$$x + c_1'(y) = x \Rightarrow c_1'(y) = 0 \Rightarrow c_1(y) = c_2$$

$$\Rightarrow f(x,y) = xy + c_2$$

$$\int_{(-1,1)}^{(1,1)} x\,dy + y\,dx = xy + c_2\Big]_{(-1,1)}^{(1,1)}$$

$$= (1)(1) - (-1)(1) = 2$$

(b) $\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dv$

$= \iiint_Q (1 - 1 + 1)\,dv = \iiint_Q 1\,.dv$, using spherical coordinates

$$\text{Flux} = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\rho^3 \sin\phi\Big]_0^1 d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}\cos\phi\Big]_0^{\pi} d\theta = \int_0^{2\pi} \frac{2}{3}\,d\theta = \frac{2}{3}\theta\Big)_0^{2\pi} = \frac{4\pi}{3}$$

(c) $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl} F) \cdot \vec{n} \, ds$

L.H.S: $\oint_C \vec{F} \cdot d\vec{r} = \oint_C \langle x, -y, z \rangle \cdot \langle dx, dy, dz \rangle$

$= \oint_C (x\,dx - y\,dy + z\,dz)$

C: $x = 3\cos\theta$, $y = 3\sin\theta$, $z = 0$, $0 \le \theta \le 2\pi$

$dx = -3\sin\theta\,d\theta$, $dy = 3\cos\theta\,d\theta$, $dz = 0$

$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 3\cos\theta(-3\sin\theta)\,d\theta - 3\sin\theta \cdot 3\cos\theta\,d\theta + 0$

$= -9\int_0^{2\pi} (2\cos\theta\sin\theta)\,d\theta = -\frac{18}{2} \sin^2\theta \Big]_0^{2\pi} = 0$

R.H.T: $\text{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & z \end{vmatrix} = 0 \cdot \vec{i} - 0 \cdot \vec{j} + 0 \cdot \vec{k}$

$\Rightarrow \iint_S (\text{curl} F) \cdot \vec{n} \, ds = 0$

$\Rightarrow$ L.H.S = R.H.S