King Saud University,
Department of Mathematics.

M203 (Differential and integral Calculus),
Final Examination/ Second Sem. 1446

Max. Marks: 40 Marks: [Q1) 4+4+4, Q2) 4+4+4, Q3) 4+4+4+4] Time: **3 hs.**

Q 1. (a) Find the sum of the series $\sum_{n=0}^{\infty} \frac{5 \cdot 2^{3n+1} - 2 \cdot 3^{2n+1}}{12^n}$, if it exists.

(b) Find the interval of convergence and the radius of convergence of the power series: $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n \cdot 3^n}$.

(c) Find the Maclaurin series of the function $f(x) = e^x$ and use it to obtain a Maclaurin series of the function xe^{3-2x}.

Q 2. (a) By reversing the order, evaluate the double integral

$$\int_0^2 \int_{y^2}^4 y \cos(x^2) dx dy.$$

(b) Find the mass of the lamina that has the shape of the region bounded by the graphs of the equations $y = 4 - x^2$, $y = 0$, and density at any point (x, y) is directly proportional to the distance between (x, y) and the x-axis.

(c) Evaluate the triple integral by changing it to spherical coordinates:

$$\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} y\sqrt{x^2 + y^2 + z^2} dz dy dx.$$

Q 3. (a) Find the work done by the vector force $\vec{F}(x, y) = y \vec{i} + x \vec{j}$ in moving a particle from $(1, -1)$ to $(1, 1)$ along the curve $x = y^2$.

(b) Show that the following line integral is independent of path and find its value: $\int_{(0,1)}^{(1,0)} [2y^3 \cos(x) + e^x - 3]dx + [6y^2 \sin(x) + 3e^y - 5]dy$.

(c) Find the flux of the force $\vec{F}(x, y, z) = 2\vec{i} + 3\vec{j} + z\vec{k}$ through the surface S of the solid bounded by the graphs of $z = x^2+y^2$ and $z = 2-x^2-y^2$.

(d) Verify Stokes theorem for the force $\vec{F}(x, y, z) = x\vec{i} - y\vec{j} + z\vec{k}$ and the surface S that is the portion of the paraboloid $z = x^2+y^2$ with boundary curve C having parametric equations $x = \cos(t), y = 1 + \sin(t)$, $z = 2 + 2\sin(t)$, $0 \le t \le 2\pi$.

King Saud University

Department of Mathematics

M-203 (Diff. and Integral Calculus)

Final Examination (II Semester 1446
2024/2025)

Max. Marks: 40 Time: 3 Hours

Solution to the Questions

Q#1(a) Find the Sum of the Series $\sum_{n=0}^{\infty} \frac{5\cdot 2^{3n+1} - 2\cdot 3^{2n+1}}{12^n}$.

[Marks: 4]

Soln. $\sum_{n=0}^{\infty} \frac{5\cdot 2^{3n+1}}{12^n} - \sum_{n=0}^{\infty} \frac{2\cdot 3^{2n+1}}{12^n}$ 1

$= 10\sum_{n=0}^{\infty} \frac{2^{3n}}{12^n} - 6\sum_{n=0}^{\infty} \frac{3^{2n}}{12^n} = 10\sum_{n=0}^{\infty} \frac{8^n}{12^n} - 6\sum_{n=0}^{\infty} \frac{9^n}{12^n}$

$= 10\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - 6\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = 10\left[\frac{1}{1-\frac{2}{3}}\right] - 6\left[\frac{1}{1-\frac{3}{4}}\right]$

$= \underset{1}{10(3)} - \underset{1}{6(4)} = 30 - 24 = \underline{\underline{6}}$ 1

Q#1(b) Find the Interval of Convergence and radius of Convergence of the power Series: $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n\cdot 3^n}$.

[Marks: 4]

Soln. We apply Absolute Ratio-test:

$\lim_{n\to\infty} \left| \frac{(2x+1)^{n+1}}{(n+1)\cdot 3^{n+1}} \times \frac{n\cdot 3^n}{(2x+1)^n} \right|$

$= \frac{1}{3}|2x+1| \iff |2x+1| < 3$

For abs. Conv. $-3 < 2x+1 < 3 \iff -2 < x < 1$ 1

At $x = -2$, we have $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ which 1
is Conv. by AST.

At $x = 1$, we have $\sum_{n=1}^{\infty} \frac{3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is 1
divergent by p-series test (or also by Integral test)

Hence I.C: $[-2, 1)$ and radius of conv: $r = \frac{1-(-2)}{2} = \frac{3}{2}$ 1

Q#1(c) Find the Maclaurin Series of the function $f(x) = e^x$ and use it to obtain a Maclaurin Series for the function $x e^{3-2x}$. [Marks: 4]

Soln: We know the Maclaurin Series as

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots \quad (1)$$

Here $f(x) = e^x \Rightarrow f(0) = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

$\vdots$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$

$\vdots$

Substituting these values in (1), we get

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \quad (2)$$

We have $x e^{3-2x} = e^3 \cdot x \cdot e^{-2x}$

Replacing x by $(-2x)$, we get

$$e^3 \cdot x \left[1 - 2x + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots + \frac{(-2x)^n}{n!} + \cdots\right]$$

$$= e^3 \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^{n+1} \quad (2)$$

Q#2(a) Evaluate the Integral by Reversing the order of the Integral $\int_0^2 \int_{y^2}^4 y \cos x^2 \, dx \, dy$ [Marks: 4]

Soln: Reversing the order, we get $\int_0^4 \int_0^{\sqrt{x}} y \cos x^2 \, dy \, dx$ (2)

$$= \int_0^4 \left[\frac{y^2}{2}\right]_0^{\sqrt{x}} \cos x^2 \, dx = \int_0^4 \frac{x}{2} \cos x^2 \, dx =$$

$$= \frac{1}{4} \left[\sin x^2\right]_0^4 = \frac{1}{4} \sin 16 \quad (1)$$

(1)

Put $x^2 = u \Rightarrow$

$2x\,dx = du$

$= \frac{1}{4} \int \cos u \, du$

$= \frac{1}{4} \sin u$

Q#2(b) Find the mass of the lamina that has the shape of the region bounded by the graphs of the equations $y = 4-x^2$, $y = 0$, and density at any point (x,y) is directly proportional to the distance between (x,y) and the x-axis. [Marks: 4]

Soln. Mass: $m = \iint_R \delta(x,y)\,dA = \int_{-2}^{2}\int_{0}^{4-x^2} ky\,dy\,dx$ (2)

$$= k\int_{-2}^{2}\left[\frac{y^2}{2}\right]_0^{4-x^2} dx = \frac{1}{2}k\int_{-2}^{2}(4-x^2)^2 dx$$

$$= \frac{1}{2}k\int_{-2}^{2}(16-8x^2+x^4)\,dx$$

$$= k\frac{1}{\cancel{2}}\times\cancel{2}\int_0^2(16-8x^2+x^4)\,dx$$

$$= k\left[16x-8\frac{x^3}{3}+\frac{x^5}{5}\right]_0^2 \quad (1)$$

$$= k\left[16(2)-\frac{64}{3}+\frac{32}{5}\right]$$

$$\therefore m = k\,\frac{480-320+96}{15} = \frac{256}{15}k \quad (1)$$

Q#2(c) Evaluate the triple Integral by changing it to Spherical Coordinates: $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$ [Marks: 4]

Soln. $\int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{2}(\rho\sin\varphi\sin\theta)(\rho)(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta$

(3)

$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}}\left[\frac{\rho^5}{5}\right]_0^2 \sin^2\varphi\sin\theta\,d\varphi\,d\theta$$

$$= \frac{2^5}{5}\left[-\cos\theta\right]_0^{\pi}\cdot\int_0^{\pi/2}\frac{1-\cos 2\varphi}{2}\,d\varphi = \frac{2^5}{5}\cdot 2\left[\frac{1}{2}\varphi-\frac{\sin 2\varphi}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 16\pi \quad (1)$$

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Q # 3(a) Find the work done by the vector force $\vec{F}(x,y) = y\vec{i} + x\vec{j}$ in moving a particle from $(1,-1)$ to $(1,1)$ along the curve $x = y^2$ [Marks: 4]

Soln. C: Choose $y = t$, $x = t^2$; $-1 \le t \le 1$ (1)

Work done: $W = \int_C y\,dx + x\,dy$

$$= \int_{-1}^{1} t \cdot 2t\,dt + t^2\,dt = \int_{-1}^{1} 3t^2\,dt \quad (2)$$

$$= 3 \times 2 \int_0^1 t^2\,dt$$

$$= 6 \times \left[\frac{t^3}{3}\right]_0^1 = \underline{\underline{2}} \quad (1)$$

(b) Show that the following line Integral is independent of path and find its value:

$$\int_{(0,1)}^{(1,0)} (2y^3\cos(x) + e^x - 3)\,dx + (6y^2\sin(x) + 3e^y - 5)\,dy$$

[Marks: 4]

Soln. We have $M(x,y) = 2y^3\cos(x) + e^x - 3 \Rightarrow \frac{\partial M}{\partial y} = 6y^2\cos x$

and $N(x,y) = 6y^2\sin(x) + 3e^y - 5 \Rightarrow \frac{\partial N}{\partial x} = 6y^2\cos x$

$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 6y^2\cos x \Rightarrow$ The line Integral (1)

is independent of path.

Now, we find its value. We have $\vec{F} = \nabla f = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$

$\therefore f_x(x,y) = 2y^3\cos x + e^x - 3$ — (1)

$f_x(x,y) = 6y^2\sin(x) + 3e^y - 5$ — (2)

Integrating (1) w.r. to x and (2) w.r. to y, we get

$f(x,y) = 2y^3\sin(x) + e^x - 3x + g(y)$ — (3), $g(y)$ is a const. of Integration

and $f(x,y) = 2y^3\sin(x) + 3e^y - 5y + h(x)$ — (4), $h(x)$ is a const. of Integ.

From (3) and (4), we get. We have $g(y) = h(x) = c$

$f(x,y) = 2y^3\sin x + e^x + 3e^y - 3x + 5y + C$ (1)

$\ldots (1,0) \ldots (1,0) - 4 - 2e$

Q#3(c) Find the flux of the force $\vec{F} = 2\vec{i} + 3\vec{j} + z\vec{k}$ through the surface S of the solid bounded by the graphs of $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$. [Marks: 4]

Soln. We apply Divergence theorem. We have

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)dV \quad ③$$

$$= \iiint_Q (0+0+1)\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r\,dz\,dr\,d\theta \quad ③$$

$$= \int_0^{2\pi}\int_0^1 r\left[2 - r^2 - r^2\right]dr\,d\theta$$

$$= \int_0^{2\pi}\left[2\frac{r^2}{2} - 2\frac{r^4}{4}\right]_0^1 d\theta \quad ④$$

$$= 2\pi\times\left(1 - \frac{1}{2}\right) = 2\pi\times\frac{1}{2} = \underline{\underline{\pi}} \quad ①$$

Q# 3(d) Verify Stokes' theorem for the force $\vec{F}(x,y,z) = x\vec{i} - y\vec{j} + z\vec{k}$ and the surface S of the solid bounded by the portion of the paraboloid $z = x^2 + y^2$ with boundary curve C having parametric equations $x = \cos(t)$, $y = 1 + \sin(t)$, $z = 2 + 2\sin(t)$; $0 \le t \le 2\pi$ [Marks: 4]

Soln. We verify the Stokes theorem given by

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds \text{ ——— } (1)$$

Soln. First, we compute R.H.S. of (1). That is $\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$
$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$$

Therefore, $\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds = 0 \quad ②$

Now, we calculate L.H.S. of (1). That is, $\oint_C \vec{F}\cdot d\vec{r}$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C x\,dx - y\,dy + z\,dz$$

$$= \int_0^{2\pi} \cos t(-\sin t)dt - (1+\sin t)\cos t\,dt + (2+2\sin t)\,2\cos t\,dt$$

$$= \int_0^{2\pi} (-\sin t\cos t - \cos t - \sin t\cos t + 4\cos t + 4\sin t\cos t)dt$$

$$= \int_0^{2\pi} (2\sin t\cos t + 3\cos t)\,dt$$

$$= \int_0^{2\pi} (\sin 2t + 3\cos t)\,dt$$

$$= \left[-\frac{\cos 2t}{2} + 3\sin t\right]_0^{2\pi} = \underline{\underline{0}}$$

(2)