

Department of Mathematics
King Saud University
Final Examination
M-203(Differential & Integral Calculus)
First Semester (1434/1435)

Max. Marks: 40

Time: 180 minutes

Marking Scheme: Q.No:1[3+3+3+3], Q.No:2[3+3+3+3], Q.No:3[3+4+6+3]

Q. No: 1 (a) Find the sum of the series $\sum_{n=1}^{\infty} \left[\frac{1}{2^n} + \frac{1}{e^{n-1}} \right]$.

(b) Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5 + 1}}$.

(c) Find the radius of convergence and interval of convergence for the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n 4^n}.$$

(d) Give Maclaurin Series of $f(x) = \sin x$ and use its first three non-zero terms to approximate

the integral $\int_{-1}^1 \frac{\sin x}{x} dx$.

Q. No: 2 (a) Evaluate the integral $\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$.

(b) Find the surface area of portion of the sphere $x^2 + y^2 + z^2 = 1$ that lies inside the cylinder $x^2 + y^2 = y$.

(c) Evaluate the integral $\iiint_Q x dV$ where Q is the solid region bounded by the coordinate planes and the plane $x + y + z = 1$.

(d) Use cylindrical coordinates to evaluate the integral $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 (x^2 + y^2) dz dy dx$.

Q. No: 3 (a) If $\vec{F}(x, y) = \frac{y}{x} \vec{i} + \vec{j}$, show that $\int_C \vec{F} \cdot d\vec{r}$ is not independent of path and evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is given by $y = \ln x$, with $x \in [1, e]$.

(b) Use Green's theorem to evaluate the line integral $\oint_C (x - y) dx + x dy$, where C is the boundary of the region bounded by $x^2 + y^2 = 9$ with $0 \leq \theta \leq \frac{\pi}{9}$.

(c) Verify the Divergence theorem by evaluating both the surface integral and the triple integral, for the vector field $\vec{F}(x, y, z) = x \vec{i} + y \vec{j} + z \vec{k}$ and surface S , where S is the upper half of the sphere $x^2 + y^2 + z^2 = 9$.

(d) Use Stokes's theorem to evaluate $\oint_S \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z)$ is the vector force given by.

$\vec{F}(x, y, z) = z^2 \vec{i} + y \vec{j} + xz \vec{k}$ and S is hemi-sphere $z = \sqrt{4 - x^2 - y^2}$, $0 \leq z \leq 4$.

Q.1

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{e^{n-1}}$ both are g.s. with $c.r < 1$

$S = \frac{1/2}{1-1/2} + \frac{1}{1-1/e} = 1 + \frac{e}{e-1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+1}}$ using limit comparison test with c'gl series $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$

$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} \frac{1/\sqrt{n^5+1}}{1/n^{5/2}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^5}}{\sqrt{n^5+1}}$

$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+1/n^5}} = 1 \neq 0$. Both series c'ge or d'ge together

\Rightarrow given series c'st

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n4^n} \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \frac{n4^n}{(n+1)4^{n+1}} |x|$

$= \frac{1}{4} |x| \Rightarrow$ c'st if $\frac{1}{4} |x| < 1 \Rightarrow -4 < x < 4$

Convergence at $x = -4$

$\sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$

d'st

$\left(\frac{1}{2}\right)$

Convergence at $x = 4$

$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

c'st A.S.

$\left(\frac{1}{2}\right)$

Interval of c'gence

$-4 < x < 4$

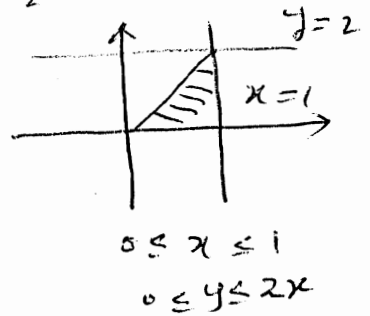
Radius " " $\rho = \frac{4 - (-4)}{2} = 4$

(2)

$$\begin{aligned} \text{Q: 2(a)} \quad \int_{-1}^1 \frac{\sin x}{x} dx &\approx \int_{-1}^1 \frac{1}{11} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \right] dx \quad (1) \\ &= \int_{-1}^1 \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} \right] dx = \left[x - \frac{x^3}{18} + \frac{x^5}{60} \right]_{-1}^1 \quad (1) \\ &= \left[1 - \frac{1}{18} + \frac{1}{60} \right] - \left[-1 + \frac{1}{18} - \frac{1}{60} \right] \\ &= 2 - \frac{1}{9} + \frac{1}{30} = \frac{5400 - 300 + 9}{2700} = \frac{5109}{2700} \\ &\approx 1.8922 \quad (1) \end{aligned}$$

$$\begin{aligned} \text{Q: 2(a)} \quad \int_0^2 \int_{y/2}^1 e^{x^2} dx dy \quad (2) \\ &= \int_0^1 \int_0^{2x} e^{x^2} dy dx = \int_0^1 e^{x^2} 2x dx \\ &= [e^{x^2}]_0^1 = e - 1 \quad (1) \end{aligned}$$

$0 \leq y \leq 2$
 $y/2 \leq x \leq 1$



$$\text{2(b)} \quad S.A. = 2 \iint_{R_{xy}} \sqrt{1 + g_x^2 + g_y^2} dA$$

$$= 2 \iint_{R_{xy}} \sqrt{1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}} dA$$

$$= 2 \iint_{R_{xy}} \frac{1}{\sqrt{1-x^2-y^2}} dA \quad (1)$$

$$\text{(1)} \quad = 2 \int_0^{\pi} \int_0^{\sin \theta} (1-r^2)^{-1/2} r dr d\theta = -\frac{1}{2} \int_0^{\pi} \left[\frac{(1-r^2)^{1/2}}{1/2} \right]_0^{\sin \theta} d\theta$$

$$= 2 - \int_0^{\pi} [\cos \theta - 1] d\theta = -2 [\sin \theta - \theta]_0^{\pi} = 2\pi \quad (1) \quad \#$$

$$z = \sqrt{1-x^2-y^2} = f(x,y)$$

$$f_x = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$f_y = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$x^2 + y^2 = y$$

$$r = \sin \theta$$

$$\boxed{0 \leq r \leq \sin \theta}$$

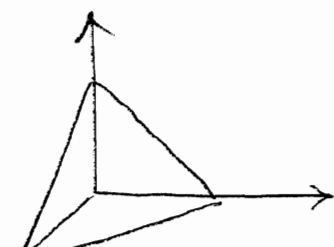
$$\boxed{0 \leq \theta \leq \pi}$$

③ ④

2c) $\iiint x \, dv$

$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz \, dy \, dx$ ① $\frac{1}{2}$ $= \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right)$

$0 \leq z \leq 1-x-y$
 $0 \leq y \leq 1-x$
 $0 \leq x \leq 1$



$= \int_0^1 \int_0^{1-x} x(1-x-y) \, dy \, dx = - \int_0^1 x \left[\frac{(1-x-y)^2}{2} \right]_0^{1-x} dx$

$= -\frac{1}{2} \int_0^1 x [(1-x-1+x)^2 - (1-x)^2] dx = +\frac{1}{2} \int_0^1 x(1-x)^2 dx$

$= \frac{1}{2} \int_0^1 (x - 2x^2 + x^3) dx = \frac{1}{2} \left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4} \right]_0^1$

$= \frac{1}{2} \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = \frac{1}{2} \left[\frac{6-8+3}{12} \right] = \frac{1}{24}$ ① $\frac{1}{24}$ \neq

2d) $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 (x^2+y^2) \, dz \, dy \, dx$ ②

$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^3 \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r^3(4-r^2) \, dr \, d\theta$

$= \int_0^{2\pi} \int_0^2 (4r^3 - r^5) \, dr \, d\theta = \int_0^{2\pi} \left[r^4 - \frac{r^6}{6} \right]_0^2 d\theta = \int_0^{2\pi} \left[16 - \frac{32}{3} \right] d\theta$

$= \frac{16}{3} (2\pi) = \frac{32\pi}{3}$ ①

(4)

Q. 3 (a) $\vec{F}(x, y) = \frac{y}{x} \vec{i} + \vec{j}$

$$\frac{\partial M}{\partial y} = \frac{1}{x} \neq \frac{\partial N}{\partial x} = 0 \quad \left(\frac{1}{2}\right)$$

Not independent of path

$C: x=t, y=ht, 1 \leq t \leq e$

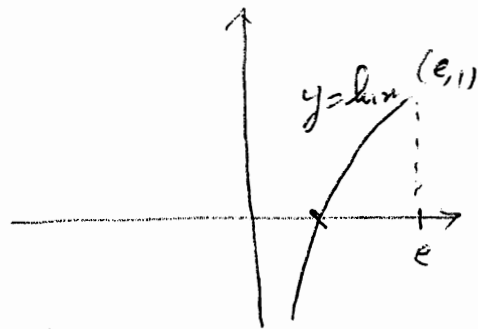
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \frac{y}{x} dx + dy$$

$$= \int_1^e \frac{ht}{t} dt + \frac{1}{t} dt$$

$$\left[\int \frac{ht}{t} dt + \int \frac{1}{t} dt \right]$$

$$= \int_1^e ht (1/t) dt + \int_1^e \frac{1}{t} dt = (ht)^2 \Big|_1^e + [ht]_1^e$$

$$= (1-0) + (1-0) = 2 \quad \left(\frac{1}{2}\right)$$



3 (b) Green's theorem $\oint_C (x-y) dx + x dy = \iint_R (1+1) dA$ (1)

$$= 2 \int_0^{\pi/9} \int_0^3 r dr d\theta = 2 \int_0^{\pi/9} \frac{9}{2} d\theta = 9(\pi/9) = \pi$$

$0 \leq r \leq 3$
 $0 \leq \theta \leq \pi/9$

(2) #

~~...~~ #

(5)

$$\textcircled{c} \quad \iint_S \vec{F} \cdot \vec{n} \, ds = \iiint_Q \left(\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial p}{\partial z} \right) dv$$

L. H. S = R. H. S

(1)

R. H. S $\iiint_Q 3 \, dv$

$$\begin{aligned} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi/2 \\ 0 \leq \rho \leq 3 \end{aligned}$$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \left[\frac{\rho^3}{3} \right]_0^3 \sin \varphi \, d\varphi \, d\theta = 27 \int_0^{2\pi} \left[-\cos \varphi \right]_0^{\pi/2} d\theta$$

$$= 27 [-0 + 1] 2\pi = 54\pi \quad \textcircled{1/2}$$

$$= 3 \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta = 3 \int_0^{2\pi} \int_0^3 \sqrt{9-r^2} \, r \, dr \, d\theta$$

Cylindrical

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 3$$

$$0 \leq z \leq \sqrt{9-r^2}$$

$$= -\frac{3}{2} \int_0^{2\pi} \left[\frac{(9-r^2)^{3/2}}{3/2} \right]_0^3 d\theta = -\frac{3}{3} \left[0 - (9)^{3/2} \right] \int_0^{2\pi} d\theta$$

$$= 27(2\pi) = 54\pi$$

d.H.S For $\vec{F} = \langle x, y, z \rangle$ and $S = \left\{ (x, y, z) \mid \begin{aligned} &x^2 + y^2 + z^2 = 9 \\ &z = \sqrt{9-x^2-y^2} \end{aligned} \right\}$

$$\iint_n \vec{F} \cdot \vec{n} \, ds = \iiint_Q \text{div}(\vec{F}) \, dv, \quad z = \sqrt{9-x^2-y^2}$$

d.H.S $\iint_n \vec{F} \cdot \vec{n} \, ds = \iint_{R_{xy}} (-Mg_x - Ng_y + P) \, dA = \iint_R \left(\frac{x^2 + y^2 + z - x^2 - y^2}{\sqrt{9-x^2-y^2}} \right) dA$

$$= \frac{9}{2} \int_0^{2\pi} \int_0^3 (9-r^2)^{1/2} (-2r) \, dr \, d\theta = -\frac{9}{2} \int_0^{2\pi} \left[\frac{(9-r^2)^{3/2}}{3/2} \right]_0^3 d\theta$$

(1)

(6)

$$= -9 \int_0^{2\pi} [0 - 3] d\theta = 27(2\pi) = 54\pi$$

(1)

$$= 54\pi$$

3 (d) $\iint_S (\text{curl } \mathbf{F}) \cdot \bar{n} \, ds$

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y & xz \end{vmatrix}$$

$$= \iint_{R_{xy}} (-Mg_x - Ng_y + P) \, dA = \iint_{R_{xy}} \left(\frac{z^2 y}{\sqrt{4-x^2-y^2}} \right) \, dA \quad (1) = \langle 0, z, 0 \rangle$$

$$= \iint_{R_{xy}} y \, dA = \int_0^{2\pi} \int_0^2 r^2 \sin \theta \, dr \, d\theta \quad (1)$$

$$z = g(x, y) = \sqrt{4-x^2-y^2}$$

$$g_y = \frac{-y}{\sqrt{4-x^2-y^2}}$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3} \right]_0^2 \sin \theta \, d\theta = \frac{8}{3} [-\cos \theta]_0^{2\pi}$$

$$= \frac{8}{3} [-1 + 1] = 0 \quad (1)$$