King Saud University, Department of Mathematics
M-203 Final Examination (Differential and Integral Calculus)
Semester-I, 1446

Max. Marks-40 Time-3 Hours

Marking Scheme: Q.1# [4+4+4], Q.2# [4+4+4], Q.3# [4+4+4+4]

Q.1 (a) Explain why the series $\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$ is convergent?

(b) Find the interval and the radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+3}}.$$

(c) Find the Maclaurin series of the function $f(x) = \ln(1+x)$, $x > -1$ and use it to find power series for the function

$$\ln\left(\frac{1+x}{1-x}\right), \; x \neq \pm 1.$$

Q.2 (a) Find the volume of the region bounded by the graph of the equation $z = e^{-x^2}$ and the planes $y = 0$, $y = x$ and $x = 1$.

(b) Find the surface area of the portion of the hemisphere $z = \sqrt{25 - x^2 - y^2}$ that lies over the region R of xy-plane bounded by the circle $x^2 + y^2 = 9$.

(c) Evaluate the integral $\iiint\limits_Q z dV$, where Q is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the planes $z = 0$, $x = \sqrt{3}y$ and $x = y$.

Q.3 (a) Find the work done by the force $\overrightarrow{F} = -yi + xj$ in moving a particle from the point $(4, 2)$ to the point $(1, 1)$ along the parabola $y^2 = x$.

(b) If C is a closed path containing the segments of the parabola $y^2 = -x+1$ and the line $y = 1 + x$, evaluate

$$\oint\limits_C (1-y)\, dx + (4+x)\, dy.$$

(c) Find the flux of the force $\overrightarrow{F} = (x - y^2)\overrightarrow{i} + (y - z^2)\overrightarrow{j} + (z - x^2)\overrightarrow{k}$ through the surface S given by $z = 4 - x^2 - y^2$, $z \geq 0$.

(d) Verify the Stokes's theorem for the surface S that is the part of the paraboloid $2z = x^2 + y^2$ with boundary C given by

$$x = 1 + \cos t, \; y = \sin t, \; z = 1 + \cos t, \quad 0 \leq t \leq 2\pi$$

and the force $\overrightarrow{F} = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$.

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King Saud University
Department of Mathematics, College of Science
M-203 (Differential and Integral Calculus)
I Semester 1446 · Final Examination
Solution to the Questions

Max. Marks: 40 Time: 3 Hours

Marking Scheme: Q#1 [4+4+4], Q#2 [4+4+4]: Q#3 [4+4+4+4]

Q#1)(a) Explain why the series $\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$ is convergent? [Marks: 4]

Soln. we know $\frac{1}{e^{n^2}} \leq \frac{1}{e^n}$ for every $n \geq 1$ ②

Since $\sum_{n=0}^{\infty} \frac{1}{e^n}$ is a convergent Geom. Series, ① $\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$

by Comparison test (BCT) is also convergent. ①

1(b) Find the interval of convergence and radius of conv. for the power series: $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+3}}$. [Marks: 4]

Soln. we apply absolute Ratio. test

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{\sqrt{(n+1)^2+3}} \times \frac{\sqrt{n^2+3}}{(-1)^n x^n} \right| = \lim_{n\to\infty} \sqrt{\frac{n^2+3}{n^2+2n+4}}\,|x|$$
$$= |x|$$

The Series will be absolutely conv. if $|x| < 1$

$\Rightarrow -1 < x < 1$ ①

At $x = -1$, we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$. By LCT comparing with $\sum \frac{1}{n}$, it is a divergent ①

At $x = 1$, we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$ which is conv. by AST ①

Hence Interval of conv: $(-1, 1)$ and radius of conv: $r = \frac{1-(-1)}{2}$
$= 1$ ①

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Q#1(C) Find the Maclaurin Series of the function $f(x) = \ln(1+x)$, $x > -1$ and use it to find the power series for the function $\ln\left(\frac{1+x}{1-x}\right)$, $x \neq \pm 1$. [Marks: 4]

Soln. $f(x) = \ln(1+x)$

$\Rightarrow f(0) = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$

$f''(x) = (-1)\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = 2\frac{1}{(1+x)^3} \Rightarrow f'''(0) = 2$

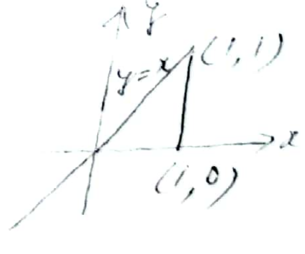
$$f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} \quad |x|<1 \quad ②$$

Hence $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \cdots \quad |x|<1$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \quad ②$$

Q#2(a) Find the volume of the region bounded by the graph of the equation $z = e^{-x^2}$ and the planes $y = 0$, $y = x$ and ~~z~~ $x = 1$ [Marks: 4]



Soln. $R = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$

$\therefore$ Volume $V = \iint_R z \, dA$

$$= \int_0^1 \int_0^x e^{-x^2} dy\, dx \quad ② \quad = \int_0^1 e^{-x^2} [y]_0^x dx$$

$$= \int_0^1 x e^{-x^2} dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$

$$= -\frac{1}{2}(e^{-1} - 1) = \underline{\underline{-\frac{1}{2}\left(\frac{e-1}{e}\right)}} \quad ②$$

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Q # 2(b) Find the surface area of the portion of the hemisphere $z = \sqrt{25 - x^2 - y^2}$ that lies over the region R of the xy - plane bounded by the circle $x^2 + y^2 = 9$.

[Marks: 4]

Soln. We have $z = \sqrt{25 - x^2 - y^2} = g(x, y)$

$$g_x = \frac{x}{\sqrt{25 - x^2 - y^2}} \; ; \; g_y = \frac{y}{\sqrt{25 - x^2 - y^2}}$$

$$\sqrt{1 + g_x^2 + g_y^2} = \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}}$$

$$= \sqrt{\frac{25 - x^2 - y^2 + x^2 + y^2}{25 - x^2 - y^2}} = \frac{5}{\sqrt{25 - x^2 - y^2}} \quad ①$$

$$\therefore S.A = \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} \, r \, dr \, d\theta \quad ②$$

Put $25 - r^2 = u \Rightarrow$

$-2r\,dr = du$

$\therefore r\,dr = -\frac{1}{2} du$

$$= 2\pi \left[-5\sqrt{25 - r^2} \right]_0^3$$

$-\frac{5}{2} \int \frac{1}{\sqrt{u}} du$

$$= -10\pi (4 - 5) = \underline{\underline{10\pi}} \quad ①$$

$= -\frac{5}{2} \times 2 u^{\frac{1}{2}}$

Q # 2(c) Evaluate the Integral $\iiint_Q z \, dV$, where Q is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the planes $z = 0$, $x = \sqrt{3} y$, and $x = y$.

[Marks: 4]

Soln. We have $\iiint_Q z \, dV = \int_{\frac{\pi}{6}}^{\pi/4} \int_0^2 \int_0^{\sqrt{4 - r^2}} z \, dz \, r \, dr \, d\theta$ ③

$x = y$

$r \cos\theta = r \sin\theta$

$\therefore \tan\theta = 1 = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

$x = \sqrt{3} y$

$r \cos = \sqrt{3} \sin\theta$

$\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\therefore \theta = \pi/6$

$4 - r^2 = u \Rightarrow$

$-2r\,dr = du$

$\therefore r\,dr = -\frac{1}{2} du$

$$= 2 \int_{\pi/6}^{\pi/4} \int_0^2 \left[\frac{z^2}{2} \right]_0^{\sqrt{4 - r^2}} = 2 \int_{\pi/6}^{\pi/4} \int_0^2 r(4 - r^2) dr \, d\theta$$

$$= + 4 \int_{\pi/6}^{\pi/4} d\theta = 4 \left[\frac{\pi}{4} - \frac{\pi}{6} \right]$$

$$= 4 \left(\frac{3\pi - 2\pi}{3 \cdot 12} \right) = \underline{\underline{\frac{\pi}{3 \times 2}}} \quad ①$$

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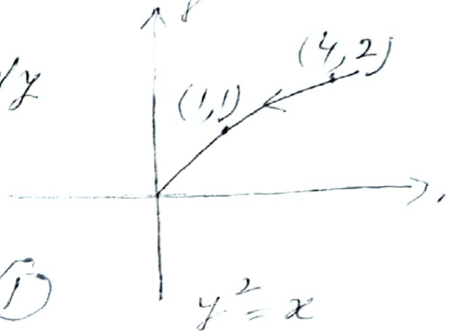
Q# 3(a) Find the work done by the force $\vec{F} = -y\vec{i} + x\vec{j}$ in moving a particle from the point (4,2) to the point (1,1) along the parabola $y^2 = x$ [Marks: 4]



Soln. Work done: $W = \oint_C M dx + N dy$

$= \oint_C -y dx + x dy$ ①

$= \int_{2}^{1} -t \cdot 2t dt + t^2 dt$ C: Choose $y = t$, $x = t^2$; $1 \le t \le 2$ ②

$= -\int_1^2 (-2t^2 + t^2) dt$

$= -\int_1^2 -t^2 dt = -\left[-\frac{t^3}{3}\right]_1^2$

$= -\left[-\frac{8}{3} + \frac{1}{3}\right] = -\left(-\frac{7}{3}\right)$

$= \frac{7}{3}$ ①

Alternate method:

C: $x = (3-t)^2$, $y = 3-t$; $1 \le t \le 2$ ②

$W = \int \vec{F} \cdot d\vec{r} = \int_1^2 (3-t)^2 dt$ ①

$= \int_1^2 (9 - 6t + t^2) dt$

$= \left[9t - \overset{3}{\cancel{6}} \frac{t^2}{\cancel{2}} + \frac{t^3}{3}\right]_1^2$

$= \cancel{18} - \cancel{12} + \frac{8}{3} - \cancel{9} + \cancel{3} - \frac{1}{3}$

$= \frac{8 - 1}{3} = \frac{7}{3}$ ①

Q#3(b) If C is a closed path containing the segments of the parabola $y^2 = -x+1$ and the line $y = 1+x$, evaluate $\oint_C (1-y)dx + (4+x)dy$. [Marks: 4]

Soln. We have $y^2 = -x+1$; $y = 1+x$

or, $x = 1-y^2$; $x = y-1$

Hence, we have $1-y^2 = y-1$

or, $-y^2 - y + 2 = 0$

or, $y^2 + y - 2 = 0$

or, $y^2 + 2y - y - 2 = 0$

or, $y(y+2) - (y+2) = 0$

$\therefore (y+2)(y-1) = 0$

$\therefore y = -2$ or $y = 1$

We evaluate $\oint_C (1-y)dx + (4+x)dy$ ①

Applying Green's therem, we have

$$2\int dxdy = 2\int_{-2}^{1}\int_{y-1}^{1-y^2} dxdy \quad ②$$

$$= 2\int_{-2}^{1} (1-y^2-y+1)dy = \underline{\underline{9}} \quad ①$$

Q# 3(c) Find the flux of the force $\vec{F} = (x-y^2)\vec{i} + (y-x^2)\vec{j} + (z-x^2)\vec{k}$

through the surface S given by $z = 4-x^2 y^2$, $z \geq 0$. [Marks: 4]

Soln. As S is a closed surface, we have

$$\text{Flux} = \iiint_Q \nabla\cdot\vec{F}\,dV \overset{\text{By Divergence th.}}{=} \iiint_Q \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)dV$$

$$= 3\iiint_Q dV = 3\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\,dz\,dr\,d\theta \quad ③$$

$$= 3\int_0^{2\pi}\int_0^2 r[4-r^2)\,dr\,d\theta = 3\int_0^{2\pi}\left[4\frac{r^2}{2} - \frac{r^4}{4}\right]_0^2$$

$$= 3\times 2\pi(8-4) = 24\pi \quad ①$$

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Q# 3(d) Verify the Stokes's theorem for the surface S that is the part of the paraboloid $2z = x^2+y^2$ with boundary C given by

$x = 1+\cos t,\ y = \sin t,\ z = 1+\cos t,\ 0 < t \le 2\pi$

and the force $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$. [Marks: 4]

Soln. We have the Stokes's theorem

$$\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds = \oint_C \vec{F}\cdot d\vec{r} \quad —(1)$$

We ~~first~~ calculate L.H.S. of (1). That is,

$$\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds.$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \textcircled{M} & \textcircled{N} & \textcircled{P} \\ x & y & z \end{vmatrix} = 0$$

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds = 0 \quad ②$$

Now, we evaluate $\oint_C \vec{F}\cdot d\vec{r}$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \oint_C x\,dx + y\,dy + z\,dz$$

$$= \int_0^{2\pi} (1+\cos t)(-\sin t)\,dt + \sin t\cos t\,dt + (1+\cos t)(-\sin t)\,dt \quad ①$$

$$= \int_0^{2\pi} (-\sin t - \cos t\sin t + \sin t\cos t - \sin t - \sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (-2\sin t - \cos t\sin t)\,dt$$

$$= \int_0^{2\pi} (-2\sin t - \tfrac{1}{2}\sin 2t)\,dt$$

$$= \left[2\cos t + \tfrac{1}{2}\,\frac{\cos 2t}{2}\right]_0^{2\pi} = \underline{\underline{0}} \quad ①$$

Hence L.H.S. of (1) = R.H.S. of (1).

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