Journal of Geometry and Physics 217 (2025) 105644

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Some basic properties of Ricci almost solitons

Sharief Deshmukh [a,1], Nasser Bin Turki [a,1], Hemangi Madhusudan Shah [b,1], Gabriel-Eduard Vîlcu [c,d,*,1]

[a] Department of Mathematics, College of Science, King Saud University, P.O. Box-2455, Riyadh, 11451, Saudi Arabia
[b] Harish Chandra Research Institute, A CI of Homi Bhabha National Institute, Chhatnag Road, Jhusi, Prayagraj, 211019, India
[c] Department of Mathematics and Informatics, National University of Science and Technology Politehnica Bucharest, 313 Splaiul Independenței, Bucharest, 060042, Romania
[d] "Gheorghe Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Calea 13 Septembrie 13, Bucharest, 050711, Romania

ARTICLE INFO

Article history:
Received 6 March 2025
Received in revised form 17 August 2025
Accepted 8 September 2025
Available online 11 September 2025

MSC:
53C21
53C25
53E20
53E40
53B50

Keywords:
Ricci soliton
Ricci almost soliton
Generalized solitons
n-sphere
Ricci curvature

ABSTRACT

Ricci solitons are stationary solutions of a famous PDE for Riemannian metrics, known under the name of Ricci flow equation. An almost Ricci soliton is a remarkable generalization of Ricci solitons by allowing the soliton constant in Ricci flow equation to be a smooth function. In the present paper, we focuss our study on the most important class of almost Ricci solitons, namely gradient Ricci almost solitons $(M^n, g, \nabla\sigma, f)$ with potential function σ and associated function f, abbreviated as *GRRAS* $(M^n, g, \nabla\sigma, f)$. On a nontrivial *GRRAS* $(M^n, g, \nabla\sigma, f)$, these two functions σ and f together with scalar curvature τ play a significant role. Among the basic properties of a connected *GRRAS* $(M^n, g, \nabla\sigma, f)$, it has been observed that there exists a smooth function δ called the connector of the *GRRAS* $(M^n, g, \nabla\sigma, f)$ as it connects the gradients of the potential function σ and the associated function f, respectively. In our first result it is shown that a nontrivial *GRRAS* $(M^n, g, \nabla\sigma, f)$ with connector δ gives a generalized soliton, thus establishing an unexpected duality. In our second result, we show that a compact and connected nontrivial *GRRAS* $(M^n, g, \nabla\sigma, f)$ with connector $\delta = -c$, for a positive constant c, and a suitable lower bound on the integral of the Ricci curvature $Ric(\nabla\sigma, \nabla\sigma)$ is isometric to the n-sphere $S^n(c)$ and the converse too is shown to hold. In the third result it is established that a complete and simply connected nontrivial *GRRAS* $(M^n, g, \nabla\sigma, f)$ of positive scalar curvature, with a suitable lower bound on $Ric(\nabla\sigma, \nabla\sigma)$ and the vector $\nabla\sigma$ being eigenvector of the Hessian operator H_σ with an appropriate eigenvalue, gives a characterization of $S^n(c)$. In our final result, we consider a compact and connected nontrivial *GRRAS* $(M^n, g, \nabla\sigma, f)$ of positive scalar curvature and ask the associated function f to satisfy a Poison equation to get yet other characterization of $S^n(c)$.

* Corresponding author at: "Gheorghe Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Calea 13 Septembrie 13, Bucharest, 050711, Romania.

E-mail addresses: shariefd@ksu.edu.sa (S. Deshmukh), nassert@ksu.edu.sa (N. Bin Turki), hemangimshah@hri.res.in (H.M. Shah), gabriel.vilcu@upb.ro (G.-E. Vîlcu).

[1] These authors contributed equally to this work.

https://doi.org/10.1016/j.geomphys.2025.105644
0393-0440/

1. Introduction

An n-dimensional Ricci soliton is a quadruplet $(M^n, g, \mathbf{u}, \mu)$ consisting in a smooth n-dimensional manifold M^n equipped with a Riemannian metric g, a vector field $\mathbf{u}$ called potential field and a constant μ, satisfying (cf. [11])

$$\frac{1}{2}\pounds_{\mathbf{u}} g + Ric = \mu g, \tag{1}$$

where $\pounds_{\mathbf{u}} g$ is the Lie-derivative of the metric g with respect to $\mathbf{u}$, Ric is the Ricci tensor of (M^n, g). A Ricci soliton $(M^n, g, \mathbf{u}, \mu)$ is a stable solution of the Hamilton's Ricci flow (for details, see [9,11,20–22]). The origin of Ricci solitons lies in Hamilton's quest of solving the century old conjecture, namely the Poincaré conjecture, and latter it turned out that they were not only important in settling this conjecture but have huge impact in different aspects of differential geometry as well as relativity. A Ricci soliton $(M^n, g, \mathbf{u}, \mu)$ is said to be a gradient Ricci soliton if the potential field $\mathbf{u}$ is a gradient of a smooth function φ, that is, $\mathbf{u} = \nabla\varphi$. As remarkable examples of such solitons we have the Gaussian soliton, the cylindrical soliton, as well as the gradient Ricci solitons with a warped product structure (for details, see [16]). There had been a huge interest in the study of a gradient Ricci soliton $(M^n, g, \nabla\varphi, \mu)$ (cf. [1,11,12,17–19,21,24,29]). The notion of Ricci soliton was generalized to Ricci almost soliton, where the constant μ in Ricci soliton is replaced by a smooth function f (cf. [30]). Thus, an n-dimensional Ricci almost soliton $(M^n, g, \mathbf{u}, f)$ on a Riemannian manifold (M^n, g) consists of a vector field $\mathbf{u}$, called potential field and a function f, called associated function that satisfies (cf. [30])

$$\frac{1}{2}\pounds_{\mathbf{u}} g + Ric = fg \tag{2}$$

and that it is also a stable solution of the Ricci flow with different initial conditions compared to Ricci solitons (see [3,25]). We shall abbreviate an n-dimensional Ricci almost soliton $(M^n, g, \mathbf{u}, f)$ by a *RAS* $(M^n, g, \mathbf{u}, f)$. After the introduction of *RAS* $(M^n, g, \mathbf{u}, f)$, the focus of attention shifted from Ricci soliton to *RAS* $(M^n, g, \mathbf{u}, f)$ and first attempts were to find conditions under which a *RAS* $(M^n, g, \mathbf{u}, f)$ is a Ricci soliton, that is, the associated function f is a constant (cf. [4–6,14,15,21,32,33]). A *RAS* $(M^n, g, \mathbf{u}, f)$ is said to be nontrivial, if the associated function is not a constant, that is, the *RAS* $(M^n, g, \mathbf{u}, f)$ is not a Ricci soliton. If the potential field $\mathbf{u}$ is a gradient of a smooth function σ, called the potential function, that is, $\mathbf{u} = \nabla\sigma$, then we call the *RAS* $(M^n, g, \nabla\sigma, f)$ the gradient Ricci almost soliton or the *GRRAS* $(M^n, g, \nabla\sigma, f)$. The defining equation (2) for a *GRRAS* $(M^n, g, \nabla\sigma, f)$ takes the form

$$Hes(\sigma) + Ric = fg, \tag{3}$$

where $Hes(\sigma)$ is the Hessian of the function σ.

As a first example of a *GRRAS* $(S^n(c), g, \nabla\sigma, f)$, we consider the sphere $S^n(c)$ as an embedded surface in the Euclidean space E^{n+1} that has shape operator $T = -\sqrt{c}I$ and unit normal ξ. Then for the coordinate vector field $\frac{\partial}{\partial u^1}$ on E^{n+1}, where $u^1, .., u^{n+1}$ are coordinates on E^{n+1}, we have

$$\frac{\partial}{\partial u^1} = \mathbf{u} + \sigma\xi, \quad \sigma = \left\langle \frac{\partial}{\partial u^1}, \xi \right\rangle, \tag{4}$$

where $\mathbf{u}$ is the tangential projection of the coordinated vector field $\frac{\partial}{\partial u^1}$ on $S^n(c)$. Denoting the induced metric on $S^n(c)$ by g and letting ∇ the symmetric torsion free connection with respect to g, and differentiating equation (4) with respect to a vector field E on $S^n(c)$, while using fundamental equations of surface, we conclude

$$\nabla_E \mathbf{u} = -\sqrt{c}\sigma E, \quad \nabla\sigma = \sqrt{c}\mathbf{u}. \tag{5}$$

The Hessian of the function σ on using above equation is given by $Hes(\sigma) = -c\sigma g$ and the Ricci tensor of the sphere $S^n(c)$ being given by $Ric = (n-1)cg$, we conclude

$$Hes(\sigma) + Ric = fg,$$

where $f = ((n-1) - \sigma)c$. Hence, we get the *GRRAS* $(S^n(c), g, \nabla\sigma, f)$.

Another example comes from the Hyperbolic space $(\mathbb{H}^{n-1}, g_1)$ of constant curvature -1. We consider the warped product $M^n = \mathbb{R} \times_h \mathbb{H}^{n-1}$, where $h(t) = \cosh t$, defined on $\mathbb{R}$ and the warped product metric $g = dt^2 + h^2 g_1$. Then taking $\sigma(t) = \sinh t$ and $f(t) = \sigma(t) - (n-1)$, we get the *GRRAS* $(M^n, g, \nabla\sigma, f)$.

The motivation behind studying gradient Ricci almost solitons lies in their role as self-similar solutions to a perturbed version of the Ricci flow, known as the Ricci-Bourguignon flow, originally introduced in [7]. This flow has recently garnered substantial interest, with a resurgence of research beginning from [8]. In particular, the study of the Ricci-Bourguignon almost soliton and gradient Ricci-Bourguignon almost soliton on semi-Riemannian manifolds was initiated in [28].

In [26], authors generalized the notion of a Ricci soliton, by considering the following equation

$$\frac{1}{2}\pounds_{\mathbf{u}}g + \lambda Ric = \mu g + \nu\eta\otimes\eta, \tag{6}$$

on a Riemannian manifold (M^n, g), where λ, μ, ν are constants, and called a generalized Ricci soliton $\left(M^n, g, \mathbf{u}, \lambda, \mu, \nu\right)$. The generalized Ricci solitons depend on three real parameters and for particular values of these parameters they refer to known important classes in differential geometry (cf. [2]). For instance, they reduce to Ricci soliton equations, as well as they reduce to vacuum near-horizon geometry in general relativity, special cases of Einstein–Weyl equations (cf. [26]). There is a further generalization called a generalized soliton $\left(M^n, g, \mathbf{u}, \lambda, \mu, \nu\right)$ satisfying equation (6), where for a generalized soliton one requires λ, μ, ν are smooth functions (cf. [2]). If the potential field $\mathbf{u} = \nabla\sigma$, in the generalized soliton, we get the gradient generalized soliton $\left(M^n, g, \nabla\sigma, \lambda, \mu, \nu\right)$.

In this work, on a connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$, we first show the existence of a smooth function φ that satisfies $\nabla\varphi = 2f\nabla\sigma$ and this allows to define a smooth function δ on a *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ satisfying

$$\nabla f = \delta\nabla\sigma. \tag{7}$$

The function δ, as it connects the gradient of the associated function f to the gradient of the potential function σ of the *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$, is said to be the connector of *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$. Note that the relation $\nabla f = \delta\nabla\sigma$ means that the gradients of f and σ are *linearly dependent* at every point of M^n. This property is automatic in the *GRRAS* setting once δ is defined, but it is a nontrivial assumption in the broader context of generalized solitons. As we are interested in studying geometry of a *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ and in doing so, first we discover that, because of the connector defined in (7), there is a duality between a nontrivial *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ and the gradient generalized soliton $\left(M^n, g, \nabla f, \lambda, \mu, \nu\right)$. The *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ has its origin in the Ricci flow, where as the gradient generalized soliton $\left(M^n, g, \nabla f, \lambda, \mu, \nu\right)$ is randomly defined on a Riemannian manifold, and thus the above duality result in particular gives a proper background to the study of gradient generalized soliton.

We have seen in the case of *GRRAS* $\left(S^n(c), g, \nabla\sigma, f\right)$ that the associated function is $f = \left((n-1) - \sigma\right)c$. This implies $\nabla f = -c\nabla\sigma$, and therefore we conclude that the connector of the *GRRAS* $\left(S^n(c), g, \nabla\sigma, f\right)$ is given by $\delta = -c$. This naturally stimulates a question: Under what condition a compact and connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ with connector $\delta = -c$, for a positive constant c, is isometric to $S^n(c)$? In section 4, we answer this question and prove that a compact and connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ with scalar curvature τ and connector $\delta = -c$ for a positive constant c, satisfies

$$\int_{M^n} Ric(\nabla\sigma, \nabla\sigma) \geq \int_{M^n}\left[(nf + c\sigma - \tau)^2 + (n-1)c^2\sigma^2\right]$$

if and only if $\left(M^n, g, \nabla\sigma, f\right)$ is isometric to $S^n(c)$.

We also prove that a complete and simply connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ with positive scalar curvature τ satisfies

$$Ric(\nabla\sigma, \nabla\sigma) \geq -\frac{n-1}{n}g(\nabla\sigma, \nabla(\Delta\sigma)), \quad \nabla_{\nabla\sigma}\nabla\sigma = \frac{\Delta\sigma}{n}\nabla\sigma,$$

if and only if τ is a constant $n(n-1)c$ and the *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ is isometric to $S^n(c)$.

Note that the trace of the Fischer-Marsden equation (cf. [10,23]) on a Riemannian manifold $\left(M^n, g\right)$ with scalar curvature τ is given by the differential equation

$$\Delta h = -\frac{\tau}{n-1}h, \tag{8}$$

where τ being scalar curvature. Note that if a compact Riemannian manifold $\left(M^n, g\right)$ admits a nontrivial solution of the Fischer-Marsden equation, then the scalar curvature τ is a constant. However, if $\left(M^n, g\right)$ admits a solution of differential equation (8), the scalar curvature need not be a constant. In the last section of this paper, we consider a deformed form of the differential equation (8), namely

$$\Delta f = -\frac{\tau}{n-1}f + \frac{\tau^2}{n(n-1)} \tag{9}$$

satisfied by the associated function f of the compact and connected nontrivial *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ of positive scalar curvature τ. Indeed we show that (9) renders τ a constant, explicitly $n(n-1)c$, and also that (9) gives a characterization of the sphere $S^n(c)$.

We note that related approaches have recently been developed in [25,31]. In [25], the authors show that the Ahlfors Laplacian, traditionally linked to conformal geometry, provides a natural tool for proving rigidity results for Ricci almost solitons and constructing solutions to Einstein's vacuum constraint equations, while in [31], the authors establish some rigidity and classification results for complete and compact almost Ricci solitons, characterizing when they reduce to Einstein manifolds, often spheres, via geometric dynamics, energy conditions and orthogonal tensor decompositions. In contrast, our approach introduces the *connector* δ as a central methodological tool: it encapsulates the precise relation between the gradients of the potential function σ and the associated function f in a *GRRAS*, and serves as the key ingredient for establishing a duality with gradient generalized solitons. This perspective allows us to derive new characterizations of the sphere $S^n(c)$ via δ-driven integral inequalities, thus providing a unifying framework that complements and extends the above works.

2. Preliminaries

We consider an n-dimensional *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$, where σ and f are smooth functions on M^n satisfying

$$Hes(\sigma) + Ric = fg, \tag{10}$$

where Ric is the Ricci tensor and the Hessian $Hes(\sigma)$ is defined by

$$Hes(\sigma)(E, F) = g\left(\nabla_E \nabla\sigma, F\right), \quad E, F \in \Phi\left(M^n\right), \tag{11}$$

where $\Phi\left(M^n\right)$ is the space of smooth vector fields on M^n and ∇_E is the covariant derivative with respect to E. Note that the Hessian operator H_σ of the function σ is defined by

$$H_\sigma(E) = \nabla_E \nabla\sigma, \quad E \in \Phi\left(M^n\right), \tag{12}$$

which is related to $Hes(\sigma)$ by

$$Hes(\sigma)(E, F) = g\left(H_\sigma(E), F\right), \quad E, F \in \Phi\left(M^n\right). \tag{13}$$

Note that the Trace of the Hessian operator H_σ of the function σ is $\Delta\sigma$ the Laplacian of σ. Ricci operator S is a symmetric tensor related to Ricci tensor by

$$Ric\left(E, F\right) = g\left(S\left(E\right), F\right), \quad E, F \in \Phi\left(M^n\right)$$

and the Trace of the Ricci operator S, is the scalar curvature τ. For a local orthonormal frame $\{X_1, .., X_n\}$ on the *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$, we have the following well known expression for the gradient of the scalar curvature

$$\frac{1}{2}\nabla\tau = \sum_k \left(\nabla S\right)\left(X_k, X_k\right), \tag{14}$$

where

$$\left(\nabla S\right)(X, Y) = \nabla_X SY - S\left(\nabla_X Y\right).$$

Using the following expression for the curvature tensor field R of the *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$,

$$R\left(E, F\right)K = \left[D_E, D_F\right]K - D_{[E.F]}K, \quad E, F, K \in \Phi\left(M^n\right),$$

and the equation (12), we get

$$R\left(E, F\right)\nabla\sigma = \left(\nabla H_\sigma\right)(E, F) - \left(\nabla H_\sigma\right)(F, E), \quad E, F \in \Phi\left(M^n\right). \tag{15}$$

Lemma 1. *On an n-dimensional GRRAS $\left(M^n, g, \nabla\sigma, f\right)$ with scalar curvature τ, the following hold*

$$(i)\ \Delta\sigma = (nf - \tau), \quad (ii)\ S\left(\nabla\sigma\right) = -(n-1)\nabla f + \frac{1}{2}\nabla\tau.$$

Proof. (i) follows on taking the Trace in the equation (10). The equation (10) implies

$$H_\sigma = fI - S$$

and taking the derivative in above equation enables us to reach

$$\left(\nabla H_\sigma\right)(E, F) = E(f)F - \left(\nabla S\right)(E, F), \quad E, F \in \Phi\left(M^n\right).$$

Thus, on employing above equation in equation (15), we conclude

$$R\left(E, F\right)\nabla\sigma = E(f)F - F(f)E - \left(\nabla S\right)(E, F) + \left(\nabla S\right)(F, E), \quad E, F \in \Phi\left(M^n\right).$$

Now, chose a local frame $\{X_1, .., X_n\}$ on the *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ and use it with the above equation, in computing

$$\begin{aligned} Ric\,(F, \nabla\sigma) &= \sum_k g\,(R\,(X_k, F)\,\nabla\sigma, X_k) \\ &= -(n-1)F(f) + \sum_k g\,((\nabla S)\,(F, X_k) - (\nabla S)\,(X_k, F), X_k)\,, \end{aligned}$$

which on using symmetry of the Ricci operator S and equation (14) implies

$$Ric\,(F, \nabla\sigma) = -(n-1)F(f) + F(\tau) - \frac{1}{2}F(\tau).$$

This proves (ii). □

On a Riemannian manifold $\left(M^n, g\right)$, the squared length $\|S\|^2$ is defined by

$$\|S\|^2 = \sum_k g\,(SX_k, SX_k) \tag{16}$$

and the squared length $|Ric|^2$ is given by

$$|Ric|^2 = \sum_{jk} \left(Ric\left(X_j, X_k\right)\right)^2.$$

Thus, we have

$$\begin{aligned} \left|Ric - \frac{\tau}{n}g\right|^2 &= \sum_{jk} \left(Ric\left(X_j, X_k\right) - \frac{\tau}{n}g\left(X_j, X_k\right)\right)^2 \\ &= \sum_{jk} g\left(SX_j, X_k\right)^2 - 2\frac{\tau}{n}\sum_k Ric\,(X_k, X_k) + \frac{\tau^2}{n} \\ &= \|S\|^2 - \frac{\tau^2}{n}. \end{aligned} \tag{17}$$

Now, we state the following Lemma (cf. [6], Theorem 1)

Lemma 2. [6] *On an n-dimensional compact GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ *with scalar curvature* τ*, the following hold*

$$\int_{M^n} \left|Ric - \frac{\tau}{n}g\right|^2 = \frac{n-2}{2n}\int_{M^n} g\,(\nabla\sigma, \nabla\tau)\,.$$

3. Some basic properties of *GRRAS*

In this section, we consider an n-dimensional *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$, with potential function σ and associated function f and derive several basic properties. First we prove the following:

Lemma 3. *On an n-dimensional connected GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$*, there exists a smooth function* φ *satisfying*

$$\nabla\varphi = 2f\nabla\sigma.$$

Proof. Let τ be the scalar curvature of the *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$. Then define a smooth function $\varphi : M^n \to R$ by

$$\varphi = \tau + \|\nabla\sigma\|^2 - 2(n-1)f.$$

For $E \in \Phi\left(M^n\right)$, we have

$$E\,(\varphi) = E(\tau) + 2g\,(H_\sigma E, \nabla\sigma) - 2(n-1)E(f).$$

Now, using $H_\sigma = fI - S$ as outcome of equation (10) in above equation, we reach at

$$E\,(\varphi) = E(\tau) + 2g\,(fE - SE, \nabla\sigma) - 2(n-1)E(f),$$

that is,

$$\nabla\varphi = \nabla\tau + 2f\nabla\sigma - 2S(\nabla\sigma) - 2(n-1)\nabla f.$$

Employing (ii) of Lemma 1 in above equation, we get the result. □

Lemma 4. *On an n-dimensional connected GRRAS $(M^n, g, \nabla\sigma, f)$, there exists a smooth function δ satisfying*

$$\nabla f = \delta\nabla\sigma.$$

Proof. Note that by Lemma 3, the 1-form α on the *GRRAS* $(M^n, g, \nabla\sigma, f)$ defined by $\alpha = 2fd\sigma$ is closed and therefore, we conclude $df \wedge d\sigma = 0$ and it is equivalent to

$$E(f)F(\sigma) = F(f)E(\sigma), \quad E, F \in \Phi(M^n),$$

that is,

$$E(f)\nabla\sigma = E(\sigma)\nabla f, \quad E, F \in \Phi(M^n).$$

Choosing $E = \nabla\sigma$, we have

$$g(\nabla\sigma, \nabla f)\nabla\sigma = \|\nabla\sigma\|^2\nabla f,$$

which on taking the inner product with ∇f, yields

$$g(\nabla\sigma, \nabla f)^2 = \|\nabla\sigma\|^2\|\nabla f\|^2.$$

This proves that the vector fields ∇f and $\nabla\sigma$ are parallel and therefore, there exists a smooth function δ such that

$$\nabla f = \delta\nabla\sigma. \quad \square$$

Definition 1. *On an n-dimensional connected GRRAS $(M^n, g, \nabla\sigma, f)$ the smooth function δ of Lemma 4 is called the connector of the GRRAS $(M^n, g, \nabla\sigma, f)$.*

Recall that Ricci solitons as well as Ricci almost solitons arise as solutions of Ricci flow, where as generalized Ricci solitons as well as generalized solitons are defined using equations generalizing equations of Ricci solitons and Ricci almost solitons. However, owing to the property of *GRRAS* $(M^n, g, \nabla\sigma, f)$, that it has connector δ, gives a duality with generalized Ricci soliton, giving it background of Ricci flow. Indeed, our next result establishes this duality.

Theorem 1. *For an n-dimensional connected GRRAS $(M^n, g, \nabla\sigma, f)$ with connector δ, $(M^n, g, \nabla f, d\sigma, \delta, \delta f, \mu)$ is a gradient generalized soliton, where the function μ satisfies $d\delta = \mu d\sigma$.*

Proof. Employing Lemma 4, we have for $E \in \Phi(M^n)$

$$H_f(E) = E(\delta)\nabla\sigma + \delta H_\sigma(E), \quad E \in \Phi(M^n),$$

that is,

$$Hes(f)(E, F) = E(\delta)F(\sigma) + \delta Hes(\sigma)(E, F), \quad E, F \in \Phi(M^n).$$

Inserting equation (10), we have

$$(Hes(f) + \delta Ric)(E, F) = \delta f g(E, F) + E(\delta)F(\sigma), \quad E, F \in \Phi(M^n). \tag{18}$$

Note that by Lemma 4, we have $df = \delta d\sigma$, that is, $d\delta \wedge d\sigma = 0$. By the similar argument as in the proof of Lemma 4, we get there exists a smooth function μ such that $\nabla\delta = \mu\nabla\sigma$. Thus, equation (18) becomes

$$(Hes(f) + \delta Ric)(E, F) = \delta f g(E, F) + \mu E(\sigma)F(\sigma), \quad E, F \in \Phi(M^n),$$

that is

$$Hes(f) + \delta Ric = \delta f g + \mu d\sigma \otimes d\sigma. \tag{19}$$

This finishes the proof. □

Remark 1. *Observe that the converse of the above Theorem too holds provided the function* $\delta \neq 0$ *and* $\nabla f = \delta \nabla\sigma$. *Suppose* $\left(M^n, g, \nabla f, d\sigma, \delta, \delta f, \mu\right)$ *is a connected gradient generalized soliton with* $\nabla f = \delta\nabla\sigma$, *where the function* μ *satisfies* $d\delta = \mu d\sigma$ *and* $\delta \neq 0$. *Then using* $\nabla f = \delta\nabla\sigma$, *it follows that*

$$H_f(E) = E(\delta)\nabla\sigma + \delta H_\sigma(E), \quad E \in \Phi\left(M^n\right),$$

which in view of equation (19) implies

$$\delta H_\sigma(E) = -\delta S(E) + \delta f E + \mu E(\sigma)\nabla\sigma - E(\delta)\nabla\sigma, \quad E \in \Phi\left(M^n\right).$$

Thus, we have

$$\delta\left(H_\sigma(E) + \delta S(E) - fE\right) = 0,$$

where we used $d\delta = \mu d\sigma$, *that is,* $E(\delta) = \mu E(\sigma)$. *As* $\delta \neq 0$ *on connected* M^n *gives* $H_\sigma + S = fI$. *Hence, we get GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$.

4. GRRAS with constant connector

Given an n-dimensional connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ with connector δ, in this section we are interested in studying the geometry of a *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ with connector δ a constant. It follows from Lemma 4, that if the connector $\delta = 0$ on a connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$, then it is a Ricci soliton. Therefore, we shall focus on *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ with connector δ a nonzero constant. In particular, we prove the following characterization of a sphere:

Theorem 2. *A compact and connected nontrivial GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ *of dimension n, with connector* $\delta = -c$, *for a positive constant c, has scalar curvature* τ *and Ricci curvature* $Ric(\nabla\sigma, \nabla\sigma)$, *satisfying*

$$\int_{M^n} Ric(\nabla\sigma, \nabla\sigma) \geq \int_{M^n} \left[(nf - \tau + c\sigma)^2 + (n-1)c^2\sigma^2\right]$$

if and only if $\left(M^n, g, \nabla\sigma, f\right)$ *is isometric to* $S^n(c)$.

Proof. Recall from Definition 1 that the connector δ is the smooth function relating the gradients of f and σ via $\nabla f = \delta\nabla\sigma$. Suppose $\left(M^n, g, \nabla\sigma, f\right)$ is an n-dimensional, compact and connected nontrivial *GRRAS* with connector $\delta = -c$, for a positive constant c, having scalar curvature τ and Ricci curvature $Ric(\nabla\sigma, \nabla\sigma)$ satisfying

$$\int_{M^n} Ric(\nabla\sigma, \nabla\sigma) \geq \int_{M^n} \left[(nf - \tau + c\sigma)^2 + (n-1)c^2\sigma^2\right]. \tag{20}$$

Then by Lemma 4, we have $\nabla f = -c\nabla\sigma$, which on differentiating with respect to $E \in \Phi\left(M^n\right)$ gives

$$H_f(E) = -cH_\sigma(E)$$

and as $c > 0$, we have $H_\sigma = -\frac{1}{c}H_f$. Consequently, we compute

$$\begin{aligned}\|H_\sigma + c\sigma I\|^2 &= \left\|-\frac{1}{c}H_f + c\sigma I\right\| \\ &= \frac{1}{c^2}\left\|H_f\right\|^2 + nc^2\sigma^2 - 2\sigma\Delta f\end{aligned} \tag{21}$$

Note that using Lemma 1 and $\nabla f = -c\nabla\sigma$, we have

$$\Delta f = -c\Delta\sigma = -c(nf - \tau).$$

Thus, equation (21) assumes the form

$$\|H_\sigma + c\sigma I\|^2 = \frac{1}{c^2}\left\|H_f\right\|^2 + nc^2\sigma^2 + 2c\sigma(nf - \tau).$$

On integration, we have

$$\int_{M^n} \|H_\sigma + c\sigma I\|^2 = \frac{1}{c^2}\int_{M^n} \|H_f\|^2 + \int_{M^n} \left(nc^2\sigma^2 + 2c\sigma(nf - \tau)\right). \tag{22}$$

Next, we employ the Bochner's formula (cf. [13])

$$\int_{M^n} \left(Ric(\nabla f, \nabla f) + \|H_f\|^2 - (\Delta f)^2 \right) = 0,$$

in equation (22), arriving at

$$\int_{M^n} \|H_\sigma + c\sigma I\|^2 = \frac{1}{c^2} \int_{M^n} \left((\Delta f)^2 - Ric(\nabla f, \nabla f) \right) + \int_{M^n} \left(nc^2\sigma^2 + 2c\sigma\,(nf - \tau) \right).$$

Note that $\nabla f = -c\nabla\sigma$ and by virtue of (i) in Lemma 1, $\Delta f = c\,(nf - \tau)$, we see that above integral becomes

$$\int_{M^n} \|H_\sigma + c\sigma I\|^2 = \int_{M^n} \left[(nf - \tau + c\sigma)^2 + (n-1)c^2\sigma^2 \right] - \int_{M^n} Ric(\nabla\sigma, \nabla\sigma).$$

Using inequality (20) in above equation, we conclude

$$\int_{M^n} \|H_\sigma + c\sigma I\|^2 = 0.$$

This proves that $Hes(\sigma) = -c\sigma g$, where owing to the fact that the *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ is nontrivial implies σ is not a constant and that the constant $c > 0$. Hence, $\left(M^n, g, \nabla\sigma, f\right)$ is isometric to $S^n(c)$ (cf. [27]). Before proceeding to the proof of the reciprocal assertion, we would like to note that we have applied the Bochner formula for the function f on a compact Riemannian manifold. The compactness assumption ensures that integration yields no boundary terms, allowing us to conclude from inequality (20) that the L^2-norm in the left-hand side must vanish.

Conversely, suppose the *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ is isometric to $S^n(c)$. Then we know that by equation (5) we have the *GRRAS* $\left(S^n(c), g, \nabla\sigma, f\right)$, where $f = ((n-1) - \sigma)\,c$. This confirms that $\nabla f = -c\nabla\sigma$, that is, the connector is $\delta = -c$. Moreover, by equation (5), it follows that $\Delta\sigma = -nc\sigma$ and it implies that

$$\int_{S^n(c)} \|\nabla\sigma\|^2 = nc \int_{S^n(c)} \sigma^2 \tag{23}$$

and that the Ricci tensor of the sphere $S^n(c)$ is $Ric = (n-1)cg$. Thus, on using equation (23), we reach at

$$\int_{S^n(c)} Ric(\nabla\sigma, \nabla\sigma) = n(n-1)c^2 \int_{S^n(c)} \sigma^2. \tag{24}$$

Now, on the sphere $S^n(c)$, we have $\tau = n(n-1)c$ and $f = ((n-1) - \sigma)\,c$, and hence $nf - \tau + c\sigma = -(n-1)c\sigma$. Thus, we have

$$\int_{S^n(c)} \left[(nf - \tau + c\sigma)^2 + (n-1)c^2\sigma^2 \right] = n(n-1)c^2 \int_{S^n(c)} \sigma^2. \tag{25}$$

Through equations (24) and (25), we finish the proof. □

In the next result, we will use a nontrivial complete and connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ to find yet other characterization of the sphere $S^n(c)$. Indeed we prove the following:

Theorem 3. *An n-dimensional complete and simply connected nontrivial GRRAS $\left(M^n, g, \nabla\sigma, f\right)$, $n > 2$, of positive scalar curvature τ satisfies*

$$\text{(i) } Ric(\nabla\sigma, \nabla\sigma) \geq -\left(\frac{n-1}{n}\right) g(\nabla\sigma, \nabla\Delta\sigma), \quad \text{(ii) } H_\sigma(\nabla\sigma) = \frac{\Delta\sigma}{n}\nabla\sigma$$

if and only if τ is a constant $n(n-1)c$ and $\left(M^n, g, \nabla\sigma, f\right)$ is isometric to $S^n(c)$.

Proof. Here again, δ denotes the connector from Definition 1. Suppose an n-dimensional complete and simply connected nontrivial *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ of positive scalar curvature τ satisfies

$$Ric(\nabla\sigma, \nabla\sigma) \geq -\left(\frac{n-1}{n}\right) g(\nabla\sigma, \nabla\Delta\sigma) \tag{26}$$

and

$$H_\sigma(\nabla\sigma) = \frac{\Delta\sigma}{n}\nabla\sigma. \tag{27}$$

Now, using a local frame $\{X_1, .., X_n\}$, we compute

$$\begin{aligned}\operatorname{div}(H_\sigma(\nabla\sigma)) &= \sum_k g\left(\nabla_{X_k} H_\sigma(\nabla\sigma), X_k\right) \\ &= \sum_k g\left((\nabla H_\sigma)(X_k, \nabla\sigma) + H_\sigma(H_\sigma X_k), X_k\right).\end{aligned}$$

Employing the symmetry of the operator H_σ, we reach at

$$\operatorname{div}(H_\sigma(\nabla\sigma)) = \|H_\sigma\|^2 + g\left(\nabla\sigma, \sum_k (\nabla H_\sigma)(X_k, X_k)\right). \tag{28}$$

Also, we compute

$$\operatorname{div}((\Delta\sigma)\nabla\sigma) = g(\nabla\sigma, \nabla\Delta\sigma) + (\Delta\sigma)^2. \tag{29}$$

Now, using equations (27), (28) and (29), we conclude

$$\|H_\sigma\|^2 + g\left(\nabla\sigma, \sum_k (\nabla H_\sigma)(X_k, X_k)\right) - \frac{1}{n}g(\nabla\sigma, \nabla\Delta\sigma) - \frac{1}{n}(\Delta\sigma)^2 = 0. \tag{30}$$

On using $\Delta\sigma = TrH_\sigma$, we have for $E \in \Phi\left(M^n\right)$,

$$\begin{aligned}E(\Delta\sigma) &= \sum_k Eg(H_\sigma X_k, X_k) \\ &= \sum_k g\left((\nabla H_\sigma)(E, X_k) + H_\sigma(\nabla_E X_k), X_k\right) + \sum_k g(H_\sigma X_k, \nabla_E X_k) \\ &= \sum_k g\left((\nabla H_\sigma)(E, X_k), X_k\right) + 2\sum_k g(H_\sigma X_k, \nabla_E X_k).\end{aligned} \tag{31}$$

Note that $\nabla_E X_k = \sum_j \omega_k^j(E)X_j$, where ω_k^j are connection forms, which satisfy $\omega_k^j + \omega_j^k = 0$ and H_σ being symmetric, we have $H_\sigma X_k = \sum_i a_k^i X_i$ and $\left(a_k^i\right)$ is a symmetric matrix. Thus, we have

$$\sum_k g(H_\sigma X_k, \nabla_E X_k) = \sum_{ijk} \omega_k^j(E) a_k^i g\left(X_j, X_i\right) = 0.$$

Thus, the equation (31) becomes

$$E(\Delta\sigma) = \sum_k g\left((\nabla H_\sigma)(E, X_k), X_k\right),$$

which on using equation (15) provides

$$\begin{aligned}E(\Delta\sigma) &= \sum_k g\left((\nabla H_\sigma)(X_k, E) + R(E, X_k)\nabla\sigma, X_k\right) \\ &= \sum_k g\left(E, (\nabla H_\sigma)(X_k, X_k)\right) - Ric(E, \nabla\sigma).\end{aligned}$$

Replacing E by $\nabla\sigma$ in above equation, we have

$$\sum_k g\left(\nabla\sigma, (\nabla H_\sigma)(X_k, X_k)\right) = Ric(\nabla\sigma, \nabla\sigma) + g(\nabla\sigma, \nabla\Delta\sigma).$$

Inserting this equation in equation (30), we obtain

$$\|H_\sigma\|^2 + Ric(\nabla\sigma, \nabla\sigma) + g(\nabla\sigma, \nabla\Delta\sigma) - \frac{1}{n}g(\nabla\sigma, \nabla\Delta\sigma) - \frac{1}{n}(\Delta\sigma)^2 = 0. \tag{32}$$

Now, using equation (10), we have $H_\sigma = fI - S$ and it implies

$$\|H_\sigma\|^2 = nf^2 + \|S\|^2 - 2f\tau = \left(\|S\|^2 - \frac{1}{n}\tau^2\right) + \frac{1}{n}(nf-\tau)^2.$$

Consequently, using Lemma 1, we have

$$\|H_\sigma\|^2 = \left(\|S\|^2 - \frac{1}{n}\tau^2\right) + \frac{1}{n}(\Delta\sigma)^2$$

and inserting it in equation (32), we arrive at

$$\left(\|S\|^2 - \frac{1}{n}\tau^2\right) + \frac{n-1}{n}g(\nabla\sigma, \nabla\Delta\sigma) + Ric(\nabla\sigma, \nabla\sigma) = 0. \tag{33}$$

Now, using inequality (26) and the Schwartz's inequality

$$\|S\|^2 \geq \frac{1}{n}\tau^2$$

in equation (33), we conclude

$$\|S\|^2 = \frac{1}{n}\tau^2.$$

But equality in the Schwartz's inequality for $\|S\|^2$ holds precisely when $S = \frac{\tau}{n}I$. Consequently, this equation implies

$$(\nabla S)(E,F) = \frac{1}{n}E(\tau)F, \quad E, F \in \Phi\left(M^n\right).$$

Thus, we conclude

$$\sum_k (\nabla S)(X_k, X_k) = \frac{1}{n}\nabla\tau,$$

which in view of equation (14) and $n > 2$ gives $\nabla\tau = 0$, that is, τ is a constant say $n(n-1)c$. Moreover, as τ is positive, we confirm that the constant $c > 0$. Then the equation (10) with $S = \frac{\tau}{n}I$ gives

$$H_\sigma = \left(f - \frac{\tau}{n}\right)I, \tag{34}$$

and on taking derivative in above equation, implies

$$(\nabla H_\sigma)(E,F) = E(f)F, \quad E, F \in \Phi\left(M^n\right).$$

Now, inserting above equation in equation (15), we conclude

$$R(E,F)\nabla\sigma = E(f)F - F(f)E, \quad E, F \in \Phi\left(M^n\right).$$

Contracting above equation, we reach at

$$Ric(F, \nabla\sigma) = -(n-1)F(f),$$

that is

$$\frac{\tau}{n}\nabla\sigma = -(n-1)\nabla f.$$

Thus, as $\tau > 0$, we have

$$\nabla\sigma = -\frac{n(n-1)}{\tau}\nabla f.$$

Define $\overline{\sigma} = f - \frac{\tau}{n}$, and observe that as f is not a constant implies $\overline{\sigma}$ is not a constant. We have

$$\nabla\overline{\sigma} = \nabla f = -\frac{\tau}{n(n-1)}\nabla\sigma = -c\nabla\sigma,$$

which in view of (34) implies

$$H_{\overline{\sigma}} = -cH_{\rho} = -c\left(f - \frac{\tau}{n}\right)I = -c\overline{\sigma}I,$$

where c is a positive constant. Hence, $\left(M^n, g, \nabla\sigma, f\right)$ is isometric to $S^n(c)$ (cf. [27]).

Conversely, suppose the *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$, $n > 2$, is isometric to $S^n(c)$. Then by equation (5) it follows that *GRRAS* $\left(S^n(c), g, \nabla\sigma, f\right)$, where we have $f = ((n-1) - \sigma)c$ and $\Delta\sigma = -nc\sigma$. Clearly the scalar curvature $\tau = n(n-1)c$ is positive and we have $Ric(\nabla\sigma, \nabla\sigma) = (n-1)c\,\|\nabla\sigma\|^2$ and as $\nabla(\Delta\sigma) = -nc\nabla\sigma$, we get

$$Ric(\nabla\sigma, \nabla\sigma) = -\frac{n-1}{n} g(\nabla\sigma, \nabla(\Delta\sigma)) \tag{35}$$

and that using equation (5), we have

$$H_{\sigma}(E) = -c\sigma E,$$

for vector field E on $S^n(c)$. Thus, using $\Delta\sigma = -nc\sigma$, we conclude

$$H_{\sigma}(\nabla\sigma) = \frac{\Delta\sigma}{n}\nabla\sigma. \tag{36}$$

Equations (35) and (36) finish the proof. □

Remark 2. *Consider the warped product $M^n = \mathbb{R} \times_h \mathbb{H}^{n-1}$ with $h(t) = e^t$ and warped product metric $g = dt^2 + h^2 g_1$, where $(\mathbb{H}^{n-1}, g_1)$ is the Hyperbolic space of constant curvature -1. Define*

$$\sigma(t) = e^t - \frac{n-2}{6}e^{-2t}$$

and

$$f(t) = e^t - \frac{2(n-2)}{3}e^{-2t} - (n-1).$$

A direct computation shows that $(M^n, g, \nabla\sigma, f)$ is a non-compact GRRAS with connector

$$\delta(t) = 1 + \frac{(n-2)e^{-2t}}{\frac{n-2}{3}e^{-2t} + e^t},$$

which is not constant. Therefore, the manifold does not satisfy the hypotheses of the above Theorems, illustrating that the sphere characterizations obtained there rely essentially on the constancy of δ.

5. A generic differential equation on GRAS

One of the celebrated differential equations in differential geometry is the Fischer-Marsden equation on a Riemannian manifold (M^n, g), namely

$$(\Delta h)\, g + hRic = Hes(h) \tag{37}$$

and it has been subject of interest for last few decades (cf. [23]). A compact (M^n, g) admitting a nontrivial solution of differential equation (37) is known to have constant scalar curvature. The trace of equation (37) is the following differential equation

$$\Delta h = -\frac{\tau}{n-1}h. \tag{38}$$

In this section, we consider the deformed form of this differential equation, namely the Poisson equation

$$\Delta h = -\frac{\tau}{n-1}h + \frac{\tau^2}{n(n-1)} \tag{39}$$

on a compact and connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$. It is worth noting in the case of τ a constant, differential equations (38) and (39) are identical up to rescaling. It turns out that differential equation (39) plays a vital role in shaping the geometry of a compact and connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$ as seen in the following:

Theorem 4. *A compact and connected nontrivial GRRAS $\left(M^n, g, \nabla\sigma, f\right)$, of dimension $n > 2$, with positive scalar curvature τ and associated function f verifies*

$$\Delta f = -\frac{\tau}{n-1}f + \frac{\tau^2}{n(n-1)}$$

if and only if τ is a constant $n(n-1)c$ and the GRRAS $\left(M^n, g, \nabla\sigma, f\right)$ is isometric to $S^n(c)$.

Proof. The proof roadmap is the following. We first relate $\operatorname{div}(S\nabla\sigma)$ to the scalar curvature τ and Δf. We then integrate this relation and apply Lemma 2 to deduce that the manifold is Einstein. Finally, we combine this with equation (10) to determine H_σ explicitly and show that the manifold is isometric to $S^n(c)$.

We start by supposing that the associated function f of the n-dimensional compact and connected *GRRAS* $\left(M^n, g, \nabla\sigma, f\right)$, $n>2$, of positive scalar curvature τ, satisfies

$$\Delta f = -\frac{\tau}{n-1} f + \frac{\tau^2}{n(n-1)}. \tag{40}$$

Then (ii) in Lemma 1, implies

$$\operatorname{div}(S\nabla\sigma) = -(n-1)\Delta f + \frac{1}{2}\Delta\tau,$$

which in view of equation (40), changes to

$$\operatorname{div}(S\nabla\sigma) = \tau f - \frac{\tau^2}{n} + \frac{1}{2}\Delta\tau. \tag{41}$$

Next, choosing a local frame $\{X_1, .., X_n\}$ for computing

$$\operatorname{div}(S\nabla\sigma) = \sum_j g\left(\nabla_{X_j} S\nabla\sigma, X_j\right) = \sum_j g\left((\nabla S)\left(X_j, \nabla\sigma\right) + S\left(\nabla_{X_j}\nabla\sigma\right), X_j\right)$$

and using symmetry of S and equation (14), we arrive at

$$\begin{aligned}\operatorname{div}(S\nabla\sigma) &= \sum_j g\left(\nabla\sigma, (\nabla S)\left(X_j, X_j\right)\right) + \sum_j g\left(H_\sigma X_j, SX_j\right)\\ &= \frac{1}{2} g(\nabla\sigma, \nabla\tau) + \sum_j g\left(H_\sigma X_j, SX_j\right).\end{aligned}$$

Now, employing equation (10) in the form $H_\sigma = fI - S$ in the above equation, we derive

$$\operatorname{div}(S\nabla\sigma) = \frac{1}{2} g(\nabla\sigma, \nabla\tau) + f\tau - \|S\|^2$$

and combining it with equation (41), we conclude

$$\frac{1}{2} g(\nabla\sigma, \nabla\tau) + f\tau - \|S\|^2 = \tau f - \frac{\tau^2}{n} + \frac{1}{2}\Delta\tau$$

and did we reach at a crucial step

$$\frac{1}{2} g(\nabla\sigma, \nabla\tau) = \|S\|^2 - \frac{\tau^2}{n} + \frac{1}{2}\Delta\tau?$$

Yes, we did, as now we could integrate above equation and use Lemma 2, with $n>2$ to declare

$$\frac{n}{n-2}\int_{M^n}\left|Ric - \frac{\tau}{n} g\right|^2 = \int_{M^n}\left(\|S\|^2 - \frac{\tau^2}{n}\right).$$

Now applying role of the equation (17), above equation changes to

$$\frac{2}{n-2}\int_{M^n}\left|Ric - \frac{\tau}{n} g\right|^2 = 0,$$

proving that

$$Ric = \frac{\tau}{n} g, \quad S = \frac{\tau}{n} I. \tag{42}$$

Then, with similar arguments as in Theorem 3 for $n>2$, we get that τ is a positive constant say $n(n-1)c$, that is, we have $S=(n-1)cI$. Employing this information in equation (10), and using Lemma 1, we conclude

$$H_\sigma = fI - (n-1)cI = \left(f - \frac{\tau}{n}\right)I = \frac{\Delta\sigma}{n} I, \tag{43}$$

which yields

$$(\nabla H_{\sigma})(E,F)=\frac{1}{n}E(\Delta\sigma)F,\quad E,F\in\Phi\left(M^{n}\right).$$

Inserting above equation in equation (15), we have

$$R(E,F)\nabla\sigma=\frac{1}{n}\left(E(\Delta\sigma)F-F(\Delta\sigma)E\right),\quad E,F\in\Phi\left(M^{n}\right),$$

which yields

$$Ric\,(F,\nabla\sigma)=-\frac{n-1}{n}F(\Delta\sigma).$$

Thus, combining above equation with equation (42), we conclude

$$\frac{\tau}{n}\nabla\sigma=-\frac{n-1}{n}\nabla\left(\Delta\sigma\right),$$

which on using $\tau=n(n=1)c$, yields $\nabla\left(\Delta\sigma+nc\sigma\right)=0$ and therefore, on connected M^{n}, we have $\Delta\sigma+nc\sigma=\overline{c}$, where $\overline{c}$ is a constant. We have

$$\Delta\sigma=\overline{c}-nc\sigma \tag{44}$$

and on defining $\overline{\sigma}=\overline{c}-nc\sigma$, we see that $\overline{\sigma}$ is a nonconstant function as σ is being potential function of nontrivial *GRRAS* $\left(M^{n},g,\nabla\sigma,f\right)$, we conclude

$$\nabla\overline{\sigma}=-nc\nabla\sigma,\quad \Delta\overline{\sigma}=-nc\Delta\sigma.$$

Thus, for $E\in\Phi\left(M^{n}\right)$, and in view of equation (43), we have

$$H_{\overline{\sigma}}E=-ncH_{\sigma}E=-c\Delta\sigma E,$$

which on using equation (44), gives

$$H_{\overline{\sigma}}=-c\left(\overline{c}-nc\sigma\right)I=-c\overline{\sigma}I,$$

where $\overline{\sigma}$ is nonconstant function and c is a positive constant. Hence, the *GRRAS* $\left(M^{n},g,\nabla\sigma,f\right)$ is isometric to $S^{n}(c)$ (cf. [27]).

Conversely, suppose a compact and connected n-dimensional, $n>2$, *GRRAS* $\left(M^{n},g,\nabla\sigma,f\right)$ is isometric to $S^{n}(c)$. Then, we know that through equation (5), that we get the *GRRAS* $\left(S^{n}(c),g,\nabla\sigma,f\right)$ with associated function $f=((n-1)-\sigma)c$. Clearly, the scalar curvature $\tau=n(n-1)c>0$ and as $\Delta\sigma=-nc\sigma$, we get

$$\Delta f=-c\Delta\sigma=nc^{2}\sigma=nc\left(((n-1)c-f\right) \tag{45}$$

Also, we have

$$-\frac{\tau}{n-1}f+\frac{\tau^{2}}{n(n-1)}=-ncf+n(n-1)c^{2}=nc\left((n-1)c-f\right),$$

which on combining with equation (45), gives

$$\Delta f=-\frac{\tau}{n-1}f+\frac{\tau^{2}}{n(n-1)}$$

and this finishes the proof. □

Remark 3. *We note that some of our integral conditions bear a partial formal resemblance to certain inequalities appearing in [25] using Ahlfors Laplacian methods, though the derivations there use analytic tools related to the Ahlfors Laplacian. Here, the inequalities arise naturally through the geometric structure encoded in the connector δ, thus providing a complementary viewpoint.*

CRediT authorship contribution statement

Sharief Deshmukh: Study, Conception, Design of the manuscript. **Nasser Bin Turki:** Study, Conception, Design of the manuscript. **Hemangi Madhusudan Shah:** Study, Conception, Design of the manuscript. **Gabriel-Eduard Vîlcu:** Study, Conception, Design of the manuscript.

Acknowledgements

The authors would like to thank the reviewer for his valuable comments and suggestions, which contributed to improving the quality of the paper. Moreover, the authors acknowledge the financial support from the Ongoing Research Funding Program (ORF - 2025 - 413), King Saud University, Riyadh, Saudi Arabia.

Data availability

No data was used for the research described in the article.

References

[1] C.C. Aquino, A. Barros, E. Ribeiro Jr., Some applications of the Hodgede Rham decomposition to Ricci solitons, Result. Math. 60 (2011) 235–246.
[2] A.M. Blaga, B.-Y. Chen, Harmonic forms and generalized solitons, Result. Math. 79 (1) (2024) 197, Paper No. 1.
[3] A.M. Blaga, B.-Y. Chen, On conformal collineation and almost Ricci solitons, J. Geom. Phys. 207 (2025) 105354, 10 pp.
[4] A. Barros, E. Ribeiro Jr., Some characterizations for compact almost Ricci solitons, Proc. Am. Math. Soc. 140 (3) (2012) 1033–1040.
[5] A. Barros, J.N. Gomes, E. Ribeiro Jr., A note on rigidity of the almost Ricci soliton, Arch. Math. 100 (2013) 481–490.
[6] A. Barros, R. Batista, E. Ribeiro Jr., Compact almost Ricci solitons with constant scalar curvature are gradient, Monatshefte Math. 174 (2014) 29–39.
[7] J.-P. Bourguignon, Ricci Curvature and Einstein Metrics. Global Differential Geometry and Global Analysis (Berlin, 1979), Lecture Notes in Math., vol. 838, Springer, Berlin, 1981, pp. 42–63.
[8] G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza, L. Mazzieri, The Ricci-Bourguignon flow, Pac. J. Math. 287 (2) (2017) 337–370.
[9] G. Catino, L. Mazzieri, Gradient Einstein solitons, Nonlinear Anal. 132 (2016) 66–94.
[10] S.K. Chaubey, G.-E. Vîlcu, Gradient Ricci solitons and Fischer-Marsden equation on cosymplectic manifolds, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 116 (2022) 186.
[11] B. Chow, P. Lu, L. Ni, Hamilton's Ricci Flow, Graduate Studies in Mathematics, vol. 77, AMS Scientific Press, Providence, 2010.
[12] A. Derdzinski, A Myers-type theorem and compact Ricci solitons, Proc. Am. Math. Soc. 134 (2006) 3645–3648.
[13] S. Deshmukh, A. Al-Eid, Curvature bounds for the spectrum of a compact Riemannian manifold of constant scalar curvature, J. Geom. Anal. 15 (4) (2005) 589–606.
[14] S. Deshmukh, Almost Ricci solitons isometric to spheres, Int. J. Geom. Methods Mod. Phys. 16 (5) (2019) 9.
[15] S. Deshmukh, H. Alsodais, A note on almost Ricci solitons, Anal. Math. Phys. 10 (2020) 76.
[16] F. Du, J. Mao, Q. Wang, C. Wu, Eigenvalue inequalities for the buckling problem of the drifting Laplacian on Ricci solitons, J. Differ. Equ. 260 (7) (2016) 5533–5564.
[17] M. Eminenti, G. La Nave, C. Mantegazza, Ricci solitons: the equation point of view, Manuscr. Math. 127 (2008) 345–367.
[18] M. Fernández-López, E. García-Río, A remark on compact Ricci solitons, Math. Ann. 340 (2008) 893–896.
[19] M. Fernández-López, E. García-Río, Maximum principles and gradient Ricci solitons, J. Differ. Equ. 251 (1) (2011) 73–81.
[20] D. Ganguly, A. Bhattacharyya, Conformal Ricci soliton on almost co-Kaehler manifold, Adv. Math. 9 (10) (2020) 8399–8411.
[21] A. Ghosh, Ricci almost solitons satisfying certain conditions on the potential vector field, Publ. Math. (Debr.) 87 (2015) 103–110.
[22] R.S. Hamilton, The Ricci Flow on Surfaces, Mathematics and General Relativity, Contemp. Math., vol. 71, 1988, pp. 237–262.
[23] A.E. Fischer, J.E. Marsden, Manifolds of Riemannian metrics with prescribed scalar curvature, Bull. Am. Math. Soc. 80 (3) (1974) 479–484.
[24] M.A. Lone, I.F. Harry, A characterization of Ricci solitons in Lorentzian generalized Sasakian space forms, Nonlinear Anal. 239 (2024) 113443.
[25] J. Mikeš, S. Stepanov, I. Tsyganok, New applications of the Ahlfors Laplacian: Ricci almost solitons and general relativistic vacuum constraint equations, J. Geom. Phys. 209 (2025) 105414.
[26] P. Nurowski, M. Randall, Generalized Ricci solitons, J. Geom. Anal. 26 (2) (2016) 1280–1345.
[27] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Jpn. 14 (1962) 333–340.
[28] D.S. Patra, A. Ali, F. Mofarreh, Characterizations of Ricci-Bourguignon almost solitons on pseudo-Riemannian manifolds, Mediterr. J. Math. 19 (4) (2022) 176, 16 pp.
[29] P. Petersen, W. Wylie, Rigidity of gradient Ricci solitons, Pac. J. Math. 241 (2) (2009) 329–345.
[30] S. Pigola, M. Rigoli, M. Rimoldi, A.G. Setti, Ricci almost solitons, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) X (2011) 757–799.
[31] V.Y. Rovenski, S.E. Stepanov, I.I. Tsyganok, Back to almost Ricci solitons, Int. Electron. J. Geom. 16 (1) (2023) 208–214.
[32] R. Sharma, Almost Ricci solitons and K-contact geometry, Monatshefte Math. 175 (2014) 621–628.
[33] R. Sharma, S. Deshmukh, Ricci almost solitons with associated projective vector field, Adv. Geom. 22 (1) (2022) 1–8.