mathematics

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Eigenvectors of the De-Rham Operator

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Abstract: We aim to examine the influence of the existence of a nonzero eigenvector ζ of the de-Rham operator Γ on a k-dimensional Riemannian manifold (N^k, g). If the vector ζ annihilates the de-Rham operator, such a vector field is called a de-Rham harmonic vector field. It is shown that for each nonzero vector field ζ on (N^k, g), there are two operators T_ζ and Ψ_ζ associated with ζ, called the basic operator and the associated operator of ζ, respectively. We show that the existence of an eigenvector ζ of Γ on a compact manifold (N^k, g), such that the integral of $\text{Ric}(\zeta, \zeta)$ admits a certain lower bound, forces (N^k, g) to be isometric to a k-dimensional sphere. Moreover, we prove that the existence of a de-Rham harmonic vector field ζ on a connected and complete Riemannian space (N^k, g), having $\text{div}(\zeta) \neq 0$ and annihilating the associated operator Ψ_ζ, forces (N^k, g) to be isometric to the k-dimensional Euclidean space, provided that the squared length of the covariant derivative of ζ possesses a certain lower bound.

Keywords: de-Rham operator; eigenvector; k-sphere S_c^k; Ricci curvature; manifold; harmonic vector field

MSC: 53C20; 53C21; 53B50

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Citation: Bin Turki, N.; Deshmukh, S.; Vîlcu, G.-E. Eigenvectors of the De-Rham Operator. *Mathematics* **2023**, *11*, 4942. https://doi.org/10.3390/math11244942

Academic Editor: Ion Mihai

Received: 18 November 2023
Revised: 8 December 2023
Accepted: 11 December 2023
Published: 13 December 2023

1. Introduction

We abbreviate a k-dimensional Riemannian manifold $\left(N^k, g\right)$ by *k-RM* $\left(N^k, g\right)$. There are several operators acting on smooth vector fields on a *k-RM* $\left(N^k, g\right)$, $k \geq 2$, these being associated with the geometry of *k-RM* $\left(N^k, g\right)$, and among them, some are naturally associated with the metric g, namely the Ricci operator and the shape operator (in the case where *k-RM*, $\left(N^k, g\right)$ is treated as hypersurface of an ambient Riemannian manifold). An interesting observation made by Gray cf. [1]) is that the Ricci operator is of central importance when defining various classes of remarkable manifolds, the most natural being Einstein spaces. Similarly, the hypersurfaces in a Riemannian manifold are classified according to the behavior of the shape operator.

Given a smooth vector field ζ on a *k-RM* $\left(N^k, g\right)$, there is a symmetric operator T_ζ defined by

$$(\pounds_\zeta g)(X_1, X_2) = 2g(T_\zeta(X_1), X_2), \tag{1}$$

for $X_1, X_2 \in \Omega\left(N^k\right)$. Here, $\Omega\left(N^k\right)$ denotes the set of all smooth vector fields on N^k. We call this operator $T_\zeta : \Omega\left(N^k\right) \to \Omega\left(N^k\right)$, the basic operator of ζ. Obviously, the operator T_ζ is symmetric and it plays a key role in characterizing the geometry of ζ. We recall that ζ is Killing if T_ζ is the null operator, and it is well-known that Killing fields have a fundamental role in modeling the geometry of k-RM $\left(N^k, g\right)$ on which they live (cf. [2–10]). Also, we say that ζ is a conformal vector on an k-RM $\left(N^k, g\right)$ if the basic operator T_ζ is a scalar, namely $T_\zeta = \sigma I$ for smooth functions σ on N^k is called the conformal factor, and it is known that conformal vector fields play very important roles in investigating the geometry of k-RM $\left(N^k, g\right)$ on which they exist (cf. [7,11–22]).

Given a smooth vector ζ on an k-RM $\left(N^k, g\right)$, there is a second-order differential operator $\nabla^2\zeta : \Omega\left(N^k\right) \times \Omega\left(N^k\right) \to \Omega\left(N^k\right)$ defined by (cf. [23–25])

$$\left(\nabla^2\zeta\right)(X_1, X_2) = \nabla_{X_1}\nabla_{X_2}\zeta - \nabla_{\nabla_{X_1}X_2}\zeta, \quad X_1, X_2 \in \Omega\left(N^k\right),$$

where ∇ stands for the Riemannian connection on the k-RM $\left(N^k, g\right)$. Similar to Obata's differential equation (cf. [21,22,26,27]), the authors of [25] considered the differential equation

$$\left(\nabla^2\zeta\right)(X_1, X_2) + \lambda g(\zeta, X_1)X_2 = 0, \quad X_1, X_2 \in \Omega\left(N^k\right), \tag{2}$$

showing that, for $\zeta \neq 0$ on N^k, and for the positive constant λ, the above differential equation provides a necessary and sufficient condition for a complete and connected k-RM $\left(N^k, g\right)$ to be isometric to the k-dimensional sphere with a constant curvature λ. The rough Laplace operator $\Delta : \Omega\left(N^k\right) \to \Omega\left(N^k\right)$ is defined to be the trace of the operator ∇^2, that is,

$$\Delta Z = \sum_{j=1}^{k}\left(\nabla^2 Z\right)(F_j, F_j), \tag{3}$$

where $\{F_1, .., F_k\}$ is a local orthonormal frame on $\left(N^k, g\right)$. Note that, in [25], a significant relationship was stated between $\Delta\zeta$ and $\nabla^2\zeta$.

Recall that on a k-RM $\left(N^k, g\right)$, the Ricci operator $Q : \Omega\left(N^k\right) \to \Omega\left(N^k\right)$ is defined using the Ricci tensor Ric by

$$g(QX_1, X_2) = \mathrm{Ric}(X_1, X_2), \quad X_1, X_2 \in \Omega\left(N^k\right).$$

Clearly, Q is a symmetric operator (cf. [11,28,29]). In this work, we are interested in investigating the properties of the de-Rham operator Γ on an k-RM $\left(N^k, g\right)$. Recall that we have $\Gamma : \Omega\left(N^k\right) \to \Omega\left(N^k\right)$, defined by (cf. [7])

$$\Gamma = Q + \Delta, \tag{4}$$

where Δ is the rough Laplace operator on k-RM $\left(N^k, g\right)$. The de-Rham operator is used to characterize vector fields with an associated tensor T_ζ that is null; that is, it characterizes Killing vector fields. A vector field ζ on k-RM $\left(N^k, g\right)$ is said to be an eigenvector of Γ if

$$\Gamma(\zeta) = a\zeta,$$

for some constant a.

Next, we show that on the k-dimensional sphere S^k_c with a constant curvature c, there are nonzero vector fields, which are eigenvectors of the de-Rham operator Γ. Considering S^k_c as a hypersurface of the Euclidean space E^{k+1} with the unit normal ξ and the Weingarten map $-\sqrt{c}I$, a nonzero constant vector field $\mathbf{u}$ on the Euclidean space E^{k+1} makes it possible to write $\mathbf{u} = \zeta + \sigma\xi$, where $\sigma = \langle \mathbf{u}, \xi \rangle$ and ζ is the tangential projection of $\mathbf{u}$ to the sphere S_c^k. Then, we let g be the induced metric on the sphere S_c^k and ∇ be the Riemannian connection on S_c^k. By differentiating equation $\mathbf{u} = \zeta + \sigma\xi$ with respect to a vector field X on S_c^k, we obtain

$$\nabla_X \zeta = -\sqrt{c}\sigma X, \quad \nabla\sigma = \sqrt{c}\zeta, \tag{5}$$

where $\nabla\sigma$ stands for the gradient of σ. The last equation easily implies that

$$\left(\nabla^2\zeta\right)(X_1, X_2) = -\sqrt{c}X_1(\sigma)X_2, \quad X_1, X_2 \in \Omega\left(S_c^k\right)$$

and taking the trace of the preceding equation, we derive

$$\Delta\zeta = -\sqrt{c}\nabla\sigma,$$

and in view of (5), we obtain

$$\Delta\zeta = -c\zeta. \tag{6}$$

But, the Ricci tensor of the k-dimensional sphere S_c^k is expressed as $\mathrm{Ric} = (k-1)cg$, and consequently, the Ricci operator Q of S_c^k satisfies the identity

$$Q\zeta = (k-1)c\zeta. \tag{7}$$

Thus, Equations (4), (6) and (7) imply that

$$\Gamma(\zeta) = (k-2)c\zeta, \tag{8}$$

that is, ζ is an eigenvector of Γ on S_c^k. Further, as ζ is induced by a nonzero constant vector $\mathbf{u}$ of the Euclidean space E^{k+1} on the k-dimensional sphere S_c^k, it is easy to deduce that $\zeta \neq 0$.

The above remarks on the k-sphere S_c^k raise a natural question: Under what conditions is a compact and connected *k-RM* $\left(N^k, g\right)$ admitting a nonzero vector field ζ satisfying

$$\Gamma(\zeta) = (k-2)c\zeta$$

for some constant $c > 0$ isometric to S_c^k?

A vector field ξ on an *k-RM* $\left(N^k, g\right)$ with $\Gamma(\xi) = 0$ is called a de-Rham harmonic. There are many *k-RM* $\left(N^k, g\right)$ which admit de-Rham harmonic vector fields. For example, on the k-dimensional Euclidean space E^k, the vector field ξ is defined as

$$\xi = \sum_{j=1}^{k} y^j \frac{\partial}{\partial y^j}, \tag{9}$$

where $y^1, .., y^k$ are Euclidean coordinates on E^k satisfying $\Gamma(\xi) = 0$, that is, ξ is a de-Rham harmonic. Similarly, a k-dimensional Ricci soliton (N, g, ξ, λ) with the potential field ξ is also a de-Rham harmonic (cf. [11,14]). Also, a vector field ζ on an *k-RM* $\left(N^k, g\right)$ that has the basic operator $T_\zeta = 0$ is a de-Rham harmonic vector field, as the Killing vector field is a Jacobi-type vector field (cf. [14], p. 45). These considerations raise yet another question: Under what conditions is a connected and complete *k-RM* $\left(N^k, g\right)$ admitting a nonzero de-Rham harmonic vector field isometric to the Euclidean space E^k?

We answer these questions in Sections 3 and 4 of this paper. Concerning the first question, we state that any connected and compact *k-RM* $\left(N^k, g\right)$, $k > 2$ admitting a nonzero eigenvector ζ of Γ with $\Gamma(\zeta) = (k-2)c\zeta$ for some constant $c > 0$ and the integral of $\mathrm{Ric}(\zeta, \zeta)$ having a certain lower bound provides a characterization of the k-dimensional sphere S_c^k (cf. Theorem 1). Also, concerning second question, we prove that if there is a de-Rham harmonic vector field ζ on a connected and complete k-RM $\left(N^k, g\right)$, $k > 2$, with $\operatorname{div}(\zeta) \neq 0$ and such that the squared length of $\nabla\zeta$ has a certain lower bound, then $\left(N^k, g\right)$ is isometric to E^k, and the converse statement also holds (cf. Theorem 2). We would like to emphasize that many geometric characterizations of spheres and Euclidean spaces have been obtained in previous decades, the most recent being obtained through some remarkable differential equations developed by Al-Sodais and the authors of the present paper (cf. [30,31]).

2. Preliminaries

Let ζ be a smooth vector field on an k-RM $\left(N^k, g\right)$. Then, we can see, through Equation (1), that there is the basic operator T_ζ of the vector field ζ. We denote by α the 1-form dual, which is dual to ζ, i.e., $\alpha(X) = g(X, \zeta)$, and define a skew symmetric operator $\Psi_\zeta : \Omega\left(N^k\right) \to \Omega\left(N^k\right)$, called the associated operator of ζ by

$$d\alpha(X_1, X_2) = 2g(\Psi_\zeta(X_1), X_2), \quad X_1, X_2 \in \Omega\left(N^k\right). \tag{10}$$

Then, using the Riemannian connection ∇ on k-RM $\left(N^k, g\right)$, we obtain that, for any $X_1, X_2 \in \Omega\left(N^k\right)$, we have

$$\begin{aligned}
2g(\nabla_{X_1}\zeta, X_2) &= g(\nabla_{X_1}\zeta, X_2) + g(\nabla_{X_2}\zeta, X_1) + g(\nabla_{X_1}\zeta, X_2) - g(\nabla_{X_2}\zeta, X_1) \\
&= g([X_1, \zeta] + \nabla_\zeta X_1, X_2) + g([X_2, \zeta] + \nabla_\zeta X_2, X_1) \\
&\quad + X_1(g(\zeta, X_2)) - g(\zeta, \nabla_{X_1} X_2) - X_2(g(\zeta, X_1)) + g(\zeta, \nabla_{X_2} X_1) \\
&= \zeta(g(X_1, X_2)) + g([X_1, \zeta], X_2) + g([X_2, \zeta], X_1) \\
&\quad + X_1(\alpha(X_2)) - X_2(\alpha(X_1)) - \alpha([X_1, X_2]) \\
&= (\pounds_\zeta g)(X_1, X_2) + d\alpha(X_1, X_2) \\
&= 2g(T_\zeta(X_1), X_2) + 2g(\Psi_\zeta(X_1), X_2)
\end{aligned}$$

and we conclude that

$$\nabla_X \zeta = T_\zeta(X) + \Psi_\zeta(X), \quad X \in \Omega\left(N^k\right). \tag{11}$$

We define a function $\sigma : N^k \to R$ by

$$\sigma = \sum_{j=1}^{k} g(T_\zeta(F_j), F_j), \tag{12}$$

where $\{F_1, .., F_k\}$ is the local orthonormal frame on k-RM $\left(N^k, g\right)$. Then, as the associated operator Ψ_ζ of ζ is a skew symmetric operator, using Equation (11), it follows that

$$\operatorname{div}\zeta = \sigma. \tag{13}$$

Also, we define

$$\|T_\zeta\|^2 = \sum_{j=1}^{k} g(T_\zeta(F_j), T_\zeta(F_j)), \quad \|\Psi_\zeta\|^2 = \sum_{j=1}^{k} g(\Psi_\zeta(F_j), \Psi_\zeta(F_j))$$

and

$$\|\nabla\zeta\|^2 = \sum_{j=1}^{k} g\left(\nabla_{F_j}\zeta, \nabla_{F_j}\zeta\right).$$

Lemma 1. *For a smooth vector field ζ on a k-RM $\left(N^k, g\right)$,*

$$\left\|T_\zeta - \frac{\sigma}{k}I\right\|^2 = \|\nabla\zeta\|^2 - \|\Psi_\zeta\|^2 - \frac{1}{k}\sigma^2.$$

Proof. Using Equation (11), we derive

$$T_\zeta(X) - \frac{\sigma}{k}X = \nabla_X\zeta - \Psi_\zeta(X) - \frac{\sigma}{k}X, \quad X \in \Omega\left(N^k\right)$$

and by choosing a local frame $\{F_1, .., F_k\}$ on N^k, we conclude that

$$\begin{aligned} \left\|T_\zeta - \frac{\sigma}{k}I\right\|^2 &= \sum_{j=1}^k g\left(T_\zeta(F_j) - \frac{\sigma}{k}F_j, T_\zeta(F_j) - \frac{\sigma}{k}F_j\right) \\ &= \sum_{j=1}^k g\left(\nabla_{F_j}\zeta - \Psi_\zeta(F_j) - \frac{\sigma}{k}F_j, \nabla_{F_j}\zeta - \Psi_\zeta(F_j) - \frac{\sigma}{k}F_j\right) \\ &= \|\nabla\zeta\|^2 + \|\Psi_\zeta\|^2 + \frac{1}{k}\sigma^2 - 2\frac{\sigma}{k}\mathrm{div}\zeta - 2\sum_{j=1}^k g\left(\nabla_{F_j}\zeta, \Psi_\zeta(F_j)\right), \end{aligned} \tag{14}$$

where we have used the skew symmetry of Ψ_ζ. Note that as T_ζ is symmetric, while Ψ_ζ is skew symmetric, it follows that

$$\sum_{j=1}^k g(T_\zeta(F_j), \Psi_\zeta(F_j)) = 0$$

and by employing this information in Equation (14), while using Equations (11) and (13), we obtain

$$\left\|T_\zeta - \frac{\sigma}{k}I\right\|^2 = \|\nabla\zeta\|^2 + \|\Psi_\zeta\|^2 - \frac{1}{k}\sigma^2 - 2\|\Psi_\zeta\|^2$$

which proves the Lemma. □

Next, by making use of the expression of the curvature tensor field on N^k

$$R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3, \quad X_1, X_2, X_3 \in \Omega\left(N^k\right)$$

and Equation (11), we compute

$$\begin{aligned} R(X_1, X_2)\zeta = {} & (\nabla_{X_1}T_\zeta)(X_2) - (\nabla_{X_2}T_\zeta)(X_1) \\ & + (\nabla_{X_1}\Psi_\zeta)(X_2) - (\nabla_{X_2}\Psi_\zeta)(X_1), \end{aligned} \tag{15}$$

where, for an operator S, we have

$$(\nabla_{X_1}S)(X_2) = \nabla_{X_1}S(X_2) - S(\nabla_{X_1}X_2).$$

The Ricci tensor of *k-RM* $\left(N^k, g\right)$, denoted by Ric, is given as

$$\mathrm{Ric}(X_1, X_2) = \sum_{j=1}^k g(R(F_j, X_1)X_2, F_j),$$

and due to (15), we find that

$$\mathrm{Ric}(X, \zeta) = \sum_{j=1}^k g\left(\left(\nabla_{F_j}T_\zeta\right)(X) - (\nabla_X T_\zeta)(F_j) + \left(\nabla_{F_j}\Psi_\zeta\right)(X) - (\nabla_X\Psi_\zeta)(F_j), F_j\right).$$

Using the symmetry of T_ζ and the skew symmetric of Ψ_ζ in the above equation, as well as (12), we deduce that

$$\mathrm{Ric}(X,\zeta) = g\left(X, \sum_{j=1}^{k}\left(\nabla_{F_j} T_\zeta\right)(F_j)\right) - X(\sigma) - g\left(X, \sum_{j=1}^{k}\left(\nabla_{F_j}\Psi_\zeta\right)(F_j)\right). \tag{16}$$

Thus, we derive

$$Q(\zeta) = -\nabla\sigma + \sum_{j=1}^{k}\left(\nabla_{F_j} T_\zeta\right)(F_j) - \sum_{j=1}^{k}\left(\nabla_{F_j}\Psi_\zeta\right)(F_j), \tag{17}$$

where $\{F_1, .., F_k\}$ is a local frame on *k-RM* $\left(N^k, g\right)$.

Also, using Equation (11) in the second-order differential operator

$$\left(\nabla^2\zeta\right)(X_1, X_2) = \nabla_{X_1}\nabla_{X_2}\zeta - \nabla_{\nabla_{X_1}X_2}\zeta,$$

we obtain

$$\left(\nabla^2\zeta\right)(X_1, X_2) = \left(\nabla_{X_1} T_\zeta\right)(X_2) + \left(\nabla_{X_1}\Psi_\zeta\right)(X_2), \quad X_1, X_2 \in \Omega\left(N^k\right)$$

and consequently

$$\Delta\zeta = \sum_{j=1}^{k}\left(\nabla_{F_j} T_\zeta\right)(F_j) + \sum_{j=1}^{k}\left(\nabla_{F_j}\Psi_\zeta\right)(F_j). \tag{18}$$

3. Characterizing Spheres

In this section, we use an eigenvector $\zeta \neq 0$ of the de-Rham operator Γ on a compact *k-RM* $\left(N^k, g\right)$ with the corresponding eigenvalue nonzero, namely $\Gamma(\zeta) = (k-2)c\zeta$ for some constant $c > 0$ with a suitable lower bound for the integral of $\mathrm{Ric}(\zeta,\zeta)$ in order to discover a new characterization of the k-dimensional sphere S_c^k. It is worth noting that, usually, the spheres are characterized by making use of Killing vector fields and conformal vector fields due to the fact that the defining equations for these remarkable vector fields ease the study, enabling us to reach the goal. However, demanding solely that a vector field be an eigenvector of a basic operator is geometrically less convenient than the geometric conditions involved in the definition of Killing or conformal vector fields. Our following result is unique in this direction.

Theorem 1. *A vector field $\zeta \neq 0$ with the associated operator Ψ_ζ on a connected and compact k-RM $\left(N^k, g\right)$, $k > 2$ satisfies $\Gamma(\zeta) = (k-2)c\zeta$ for some constant $c > 0$, and the Ricci curvature satisfies*

$$\int_{N^k}\mathrm{Ric}(\zeta,\zeta) \geq \int_{N^k}\left[\frac{k-1}{k}(\mathrm{div}\zeta)^2 + \left\|\Psi_\zeta\right\|^2\right]$$

if and only if (N^k, g) is isometric to S_c^k.

Proof. Suppose that ζ is a nonzero vector field with the associated operator Ψ_ζ on a compact and connected *k-RM* $\left(N^k, g\right)$, $k > 2$ that satisfies $\Gamma(\zeta) = (k-2)c\zeta$ and that

$$\int_{N^k}\mathrm{Ric}(\zeta,\zeta) \geq \int_{N^k}\left[\frac{k-1}{k}(\mathrm{div}\zeta)^2 + \left\|\Psi_\zeta\right\|^2\right]. \tag{19}$$

By integrating the equation in Lemma 1, we have

$$\int_{N^k}\left\|T_\zeta-\frac{\sigma}{k}I\right\|^2=\int_{N^k}\left(\|\nabla\zeta\|^2-\|\Psi_\zeta\|^2-\frac{1}{k}\sigma^2\right). \tag{20}$$

Now, using a well-known integral formula (see [32])

$$\int_{N^k}\left(\mathrm{Ric}(\zeta,\zeta)+\frac{1}{2}|\pounds_\zeta g|^2-\|\nabla\zeta\|^2-(\mathrm{div}\zeta)^2\right)=0$$

in Equation (20), we obtain

$$\int_{N^k}\left\|T_\zeta-\frac{\sigma}{k}I\right\|^2=\int_{N^k}\left(\mathrm{Ric}(\zeta,\zeta)+\frac{1}{2}|\pounds_\zeta g|^2-(\mathrm{div}\zeta)^2-\|\Psi_\zeta\|^2-\frac{1}{k}\sigma^2\right),$$

which, after employing Equation (13), yields

$$\int_{N^k}\left\|T_\zeta-\frac{\sigma}{k}I\right\|^2=\int_{N^k}\left(\mathrm{Ric}(\zeta,\zeta)+\frac{1}{2}|\pounds_\zeta g|^2-\|\Psi_\zeta\|^2-\left(\frac{k+1}{k}\right)\sigma^2\right). \tag{21}$$

Also, by Equation (1), we derive

$$\frac{1}{2}|\pounds_\zeta g|^2=2\|T_\zeta\|^2$$

and (21) reduces to

$$\int_{N^k}\left\|T_\zeta-\frac{\sigma}{k}I\right\|^2=\int_{N^k}\left(\mathrm{Ric}(\zeta,\zeta)+2\|T_\zeta\|^2-\|\Psi_\zeta\|^2-\left(\frac{k+1}{k}\right)\sigma^2\right). \tag{22}$$

Note that

$$\begin{aligned}\left\|T_\zeta-\frac{\sigma}{k}I\right\|^2 &= \sum_{j=1}^{k}g\left(T_\zeta(F_j)-\frac{\sigma}{k}F_j,T_\zeta(F_j)-\frac{\sigma}{k}F_j\right)\\ &= \|T_\zeta\|^2+\frac{1}{k}\sigma^2-2\frac{\sigma}{k}\sum_{j=1}^{k}g(T_\zeta(F_j),F_j),\end{aligned}$$

and using (12), we derive

$$\left\|T_\zeta-\frac{\sigma}{k}I\right\|^2=\|T_\zeta\|^2-\frac{1}{k}\sigma^2.$$

Combining the preceding equation with (22) yields

$$\int_{N^k}\left\|T_\zeta-\frac{\sigma}{k}I\right\|^2=\int_{N^k}\left(\mathrm{Ric}(\zeta,\zeta)+2\left\|T_\zeta-\frac{\sigma}{k}I\right\|^2+\frac{2}{k}\sigma^2-\|\Psi_\zeta\|^2-\left(\frac{k+1}{k}\right)\sigma^2\right),$$

or, equivalently,

$$\int_{N^k}\left\|T_\zeta-\frac{\sigma}{k}I\right\|^2=\int_{N^k}\left(\left(\frac{k-1}{k}\right)\sigma^2+\|\Psi_\zeta\|^2\right)-\int_{N^k}\mathrm{Ric}(\zeta,\zeta). \tag{23}$$

By using Equation (13) in the inequality (19) and by using it in Equation (23), we conclude that

$$\int_{N^k} \left\| T_\zeta - \frac{\sigma}{k} I \right\|^2 \leq 0$$

which yields

$$T_\zeta = \frac{\sigma}{k} I. \tag{24}$$

Thus, by using a local frame $\{F_1, .., F_k\}$ on *k-RM* $\left(N^k, g\right)$ in the preceding equation, we deduce that

$$\left(\nabla_{F_J} T_\zeta\right)(F_j) = \frac{1}{k} F_j(\sigma) F_j.$$

Hence,

$$\sum_{j=1}^{k} \left(\nabla_{F_J} T_\zeta\right)(F_j) = \frac{1}{k} \nabla\sigma.$$

By inserting the above equation into Equations (17) and (18), we conclude that

$$Q(\zeta) = -\frac{k-1}{k} \nabla\sigma - \sum_{j=1}^{k} \left(\nabla_{F_J} \Psi_\zeta\right)(F_j)$$

and

$$\Delta\zeta = \frac{1}{k} \nabla\sigma + \sum_{j=1}^{k} \left(\nabla_{F_J} \Psi_\zeta\right)(F_j).$$

Thus, we find that

$$\Gamma(\zeta) = -\frac{k-2}{k} \nabla\sigma.$$

But, from the hypothesis, we have $\Gamma(\zeta) = (k-2)c\zeta$ and by virtue of the fact that $k > 2$, we conclude that

$$\nabla\sigma = -ck\zeta. \tag{25}$$

Note that $c > 0$ and ζ is a nonzero vector, so the above equation implies that σ is a non-constant function, Now, by differentiating Equation (25) while using Equations (11) and (24), we arrive at

$$\nabla_X \nabla\sigma = -ck\left(\frac{\sigma}{k} X + \Psi_\zeta(X)\right), \quad X \in \Omega\left(N^k\right).$$

Taking the inner product from the preceding equation with X and by using the skew symmetry of Ψ_ζ, we derive that

$$g(\nabla_X \nabla\sigma, X) = -c\sigma g(X, X).$$

After polarization, the above equation yields

$$g\left(\nabla_{X_1} \nabla\sigma, X_2\right) = -c\sigma g(X_1, X_2), \tag{26}$$

where $c > 0$ is a real constant, while σ is a non-constant function. Hence, by using the result of Obata (cf. [21,22,27]), from Equation (26), we see that $\left(N^k, g\right)$ is isometric to S_c^k.

Conversely, suppose that $\left(N^k, g\right)$ is isometric to S_c^k. Then, we can see that there exists a nonzero eigenvector ζ of the de-Rham operator on S_c^k with the required eigenvalue. Now, we need to show that (19) holds on S_c^k for the vector field ζ. Using Equation (5), we derive that

$$\Delta\sigma = \operatorname{div}(\nabla\sigma) = -\sqrt{c}\operatorname{div}\zeta = -kc\sigma.$$

Hence,

$$\sigma\Delta\sigma = -kc\sigma^2,$$

and this implies that

$$\int_{S_c^k} \|\nabla\sigma\|^2 = kc\int_{S_c^k} \sigma^2. \tag{27}$$

Moreover, by Equation (5), we see that

$$\operatorname{div}(\zeta) = -k\sqrt{c}\sigma \tag{28}$$

and with the vector field ζ being closed on S_c^k, we have

$$\Psi_\zeta = 0. \tag{29}$$

Using the expression for the Ricci tensor of the sphere S_c^k, we have

$$\operatorname{Ric}(\zeta,\zeta) = (k-1)c\|\zeta\|^2$$

and after inserting Equation (5), we obtain

$$\operatorname{Ric}(\zeta,\zeta) = (k-1)\|\nabla\sigma\|^2.$$

By integrating this equation and using Equations (27)–(29), we reach

$$\int_{S_c^k} \operatorname{Ric}(\zeta,\zeta) = \int_{S_c^k} \left[\frac{k-1}{k}(\operatorname{div}\zeta)^2 + \|\Psi_\zeta\|^2\right]$$

and this finishes the proof. □

4. Characterizing Euclidean Spaces

In this section, we use a de-Rham harmonic vector $\zeta \neq 0$ on a connected and complete *k-RM* $\left(N^k, g\right)$ that annihilates Ψ_ζ as well as a suitable lower bound for the squared length of $\nabla\zeta$ to find a new characterization of the k-dimensional Euclidean space E^k. Indeed, we prove the following:

Theorem 2. *A connected and complete k-RM* $\left(N^k, g\right)$, $k > 2$ *is isometric to the Euclidean space* E^k, *if and only if* (N^k, g) *admits a de-Rham harmonic vector field* $\zeta \neq 0$ *with* $\operatorname{div}(\zeta) \neq 0$, *annihilating the associated operator* Ψ_ζ, *and the length of* $\nabla\zeta$ *satisfies*

$$\|\nabla\zeta\|^2 \geq \|\Psi_\zeta\|^2 + \frac{1}{k}\sigma^2.$$

Proof. Suppose that ζ is a nonzero de-Rham harmonic vector field with $\operatorname{div}(\zeta) \neq 0$ and the associated operator Ψ_ζ on a complete and connected *k-RM* $\left(N^k, g\right)$, $k > 2$ that annihilates Ψ_ζ, that is,

$$\Psi_\zeta(\zeta) = 0 \tag{30}$$

and the length of $\nabla\zeta$ satisfies

$$\|\nabla\zeta\|^2 \geq \|\Psi_\zeta\|^2 + \frac{1}{k}\sigma^2. \tag{31}$$

Using the inequality (31) in Lemma 1, we conclude that

$$T_\zeta = \frac{\sigma}{k}I. \tag{32}$$

Hence,

$$\sum_{j=1}^{k}\left(\nabla_{F_J}T_\zeta\right)(F_j)=\frac{1}{k}\nabla\sigma$$

for a local frame $\{F_1,..,F_k\}$ on the k-RM $\left(N^k,g\right)$. Thus, Equations (17) and (18) take the forms

$$Q(\zeta)=-\frac{k-1}{k}\nabla\sigma-\sum_{j=1}^{k}\left(\nabla_{F_j}\Psi_\zeta\right)(F_j)$$

and

$$\Delta\zeta=\frac{1}{k}\nabla\sigma+\sum_{j=1}^{k}\left(\nabla_{F_J}\Psi_\zeta\right)(F_j).$$

But, by using $\Gamma(\zeta)=0$ and the last two equations, we derive that

$$\frac{k-2}{k}\nabla\sigma=0.$$

As $k>2$, the preceding equation implies that σ is nothing but a real constant, say c. Now, by combining Equations (11) and (31), we have

$$\nabla_X\zeta=\frac{c}{k}X+\Psi_\zeta(X),\quad X\in\Omega\left(N^k\right). \tag{33}$$

Define a function $\rho:N^k\to R$ by

$$\rho=\frac{1}{2}\|\zeta\|^2,$$

which, after differentiation and with the use of Equation (33), leads to

$$X(\rho)=g\left(\frac{c}{k}X+\Psi_\zeta(X),\zeta\right),\quad X\in\Omega\left(N^k\right).$$

Using Equation (30), we conclude that

$$\nabla\rho=\frac{c}{k}\zeta. \tag{34}$$

But, $\operatorname{div}(\zeta)=c\neq 0$ and $\zeta\neq 0$ imply that the function ρ is non-constant. By differentiating Equation (34), and using (33), we obtain

$$\nabla_X\nabla\rho=\frac{c}{k}\left(\frac{c}{k}X+\Psi_\zeta(X)\right),\quad X\in\Omega\left(N^k\right).$$

Taking the inner product in the preceding equation with X, since Ψ_ζ is skew symmetric, we deduce that

$$g(\nabla_X\nabla\rho,X)=ag(X,X),\quad X\in\Omega\left(N^k\right),$$

where the constant a is given by $ak^2=c^2$. Therefore, a is a nonzero constant. By polarizing the last equation, we deduce that

$$Hess(\rho)=ag,$$

where ρ is a non-constant function, and a is a nonzero constant. Hence, $\left(N^k,g\right)$ isometric to the k-dimensional Euclidean space E^k (cf. [33]).

Conversely, suppose that $\left(N^k,g\right)$ is isometric to the Euclidean space E^k. Then, ξ defined by (9) satisfies

$$\left(\nabla^2\xi\right)(X_1,X_2)=0,\quad X_1,X_2\in\Omega\left(E^k\right),$$

and thus we have

$$\Delta\xi = 0.$$

Also, as the Euclidean connection is flat, we have $Q(\xi) = 0$. Thus, $\Gamma(\xi) = 0$, that is, ξ is a nonzero vector field on E^k that annihilates the de-Rham operator. Moreover, by using Equation (9) with respect to the Euclidean connection ∇ on E^k, we have

$$\nabla_X \xi = X, \quad X \in \Omega\left(E^k\right), \tag{35}$$

which gives $\operatorname{div}(\xi) = k \neq 0$. Also, Equation (35) expresses that the vector field ξ is closed. Thus, we have

$$\Psi_\xi = 0. \tag{36}$$

Moreover, from (35), we see that the basic operator T_ξ associated with ξ, as suggested by Equation (1), satisfies

$$g(X_1, X_2) = g\left(T_\xi(X_1), X_2\right), \quad X_1, X_2 \in \Omega\left(E^k\right),$$

where g stands for the Euclidean metric on E^k. Thus, the basic operator is given by

$$T_\xi = I$$

and consequently,

$$\sigma = \sum_{j=1}^{k} g\left(T_\xi\left(F_j\right), F_j\right) = k. \tag{37}$$

Finally, the Equation (35), gives

$$\|\nabla\xi\|^2 = k. \tag{38}$$

By combining Equations (36)–(38), we obtain that ξ satisfies

$$\|\nabla\xi\|^2 = \left\|\Psi_\zeta\right\|^2 + \frac{1}{k}\sigma^2$$

and this ends the proof. □

5. Examples

Example 1. *Suppose that ζ is a Killing vector field on a Riemannian manifold $\left(N^k, g\right)$. Then, according to Equation (12), as $T_\zeta = 0$, we have*

$$\nabla_X \zeta = \Psi_\zeta(X)$$

and $\sigma = 0$. Then, Equations (17) and (18) imply that

$$Q(\zeta) = -\sum_{j=1}^{k}\left(\nabla_{F_j}\Psi_\zeta\right)\left(F_j\right)$$

and

$$\Delta\zeta = \sum_{j=1}^{k}\left(\nabla_{F_j}\Psi_\zeta\right)\left(F_j\right).$$

By combining these two equations, we realize that

$$\Gamma(\zeta) = 0,$$

that is, a Killing vector is an eigenvector of the de-Rham operator corresponding to an eigenvalue of 0.

Example 2. *Consider a k-dimensional Ricci soliton* $\left(N^k, g, \zeta, \lambda\right)$ *(cf. [11]). Then, for the potential field* ζ, $T_\zeta = \lambda I - Q$, *and the Equation (12) takes the form*

$$\nabla_X \zeta = \lambda X - Q(X) + \Psi_\zeta(X)$$

and $\sigma = n\lambda - \tau$, *where* τ *is the trace of* Q. *Thus,* $\nabla\sigma = -\nabla\tau$ *and by using*

$$\sum_{j=1}^{k} \left(\nabla_{F_j} Q\right)(F_j) = \frac{1}{2}\nabla\tau,$$

we see that Equations (17) and (18) imply that

$$Q(\zeta) = \frac{1}{2}\nabla\tau - \sum_{j=1}^{k} \left(\nabla_{F_j} \Psi_\zeta\right)(F_j)$$

and

$$\Delta\zeta = -\frac{1}{2}\nabla\tau + \sum_{j=1}^{k} \left(\nabla_{F_j} \Psi_\zeta\right)(F_j).$$

Thus, $\Gamma(\zeta) = 0$, *and thus, the potential field* ζ *is an eigenvector of the de-Rham operator corresponding to an eigenvalue of* 0.

Example 3. *Consider a k-dimensional connected Riemannian manifold* (N^k, g), $k > 2$, *that admits a nonzero concircular vector field* ζ *and that the Ricci curvature of* (N^k, g) *in the direction of* ζ *is a constant c, that is,*

$$Ric(\zeta, \zeta) = c\|\zeta\|^2. \tag{39}$$

Then, it follows that

$$\nabla_X \zeta = \rho X, \tag{40}$$

where ρ *is a smooth function. Then, it follows that* $T_\zeta = \rho I$, $\Psi_\zeta = 0$ *and* $\sigma = k\rho$, $\nabla\sigma = n\nabla\rho$. *Then, Equations (17) and (18) imply that*

$$Q(\zeta) = -(k-1)\nabla\rho$$

and

$$\Delta\zeta = \nabla\rho.$$

Thus,

$$\Gamma(\zeta) = -(k-2)\nabla\rho. \tag{41}$$

Now, using $T_\zeta = \rho I$, $\Psi_\zeta = 0$ *and Equation (15), we arrive at*

$$R(X_1, X_2)\zeta = X_1(\rho)X_2 - X_2(\rho)X_1,$$

that is,

$$R(X_1, \zeta)\zeta = X_1(\rho)\zeta - \zeta(\rho)X_1 \tag{42}$$

and as $R(X_1, \zeta)\zeta$ *is a symmetric operator, it follows that*

$$X_1(\rho)g(\zeta, X_2) = X_2(\rho)g(\zeta, X_1), \quad X_1, X_2 \in \Omega\left(N^k\right).$$

The above equation implies that

$$X(\rho)\zeta = g(\zeta, X)\nabla\rho, \quad X \in \Omega\left(N^k\right),$$

that is,

$$\zeta(\rho)\zeta = \|\zeta\|^2 \nabla\rho.$$

Thus, the vector fields ζ and $\nabla\rho$ are parallel, and therefore, there is a function f such that

$$\nabla\rho = f\zeta. \tag{43}$$

Inserting this information in Equation (42) leads to

$$R(X_1, \zeta)\zeta = f\left\{g(\zeta, X_1)\zeta - \|\zeta\|^2 X_1\right\}$$

and this equation implies that

$$Ric(\zeta, \zeta) = -(k-1)f\|\zeta\|^2.$$

Combining it with Equation (39) and as N^k is connected and $\zeta \neq 0$, we derive $-(k-1)f = c$. Hence, from Equations (41) and (42), we have

$$\Gamma(\zeta) = \frac{(k-2)}{k-1}c\zeta.$$

Hence, ζ is an eigenvector of the de-Rham operator with the eigenvalue $\frac{(k-2)}{k-1}c$. Note that such concircular vector fields exist on the warped product $S^1 \times_\theta S^{k-1}$, where θ is a positive function on the unit circle S^1.

6. Conclusions

In this work, we observed that a nonzero eigenvector of the de-Rham operator Γ on a *k-RM* $\left(N^k, g\right)$ could be used to obtain characterizations of a sphere as well as a Euclidean space. The scope of study initiated in this paper extends two of our recent works [30,31] on the characterizations of spheres and Euclidean spaces using a remarkable differential equation called the Fischer–Marsden equation, which is given by

$$(\Delta\sigma)g + \sigma \mathrm{Ric} = Hess(\sigma) \tag{44}$$

on a *k-RM* $\left(N^k, g\right)$ (cf. [34]). Recall that, as shown in [30], a compact *k-RM* $\left(N^k, g\right)$ admitting a non-trivial solution to (44) necessarily has a constant scalar curvature. In this article, we worked in a more general context, focusing not on the rough Laplace operator Δ that appears in Equation (44), but on the de-Rham operator Γ defined in Equation (4) as the sum of the Laplace operator Δ and the Ricci operator Q. Consequently, we abandoned the investigation of the Fischer–Marsden equation in this work, concentrating on the eigenvectors of the de-Rham operator, i.e., the vector fields on a *k-RM* $\left(N^k, g\right)$ satisfying the equation $\Gamma(\zeta) = a\zeta$ for a constant a. But, it is known that, on a *k-RM* $\left(N^k, g\right)$, each nonzero vector field ζ has the basic operator T_ζ and the associated operator Ψ_ζ and the smooth function $\sigma : N^k \to R$ defined by Equation (12). Using this, we proved that a compact manifold (N^k, g), $k > 2$ with a nonzero eigenvector ζ of the de-Rham operator Γ and the integral of the Ricci curvature $\mathrm{Ric}(\zeta, \zeta)$ with a suitable lower bound provide necessary and sufficient conditions for (N^k, g) to be isometric to the sphere S^k_c. We also demonstrated that a complete and connected Riemannian manifold (N^k, g) with a nonzero de-Rham harmonic vector field ζ with $\mathrm{div}(\zeta) \neq 0$ that annihilates the associated operator

Ψ_ζ and the squared length of the covariant derivative of ζ possessing a suitable lower bound give a characterization of the Euclidean space E^k.

For further investigations, it will be interesting to see whether the smooth function σ corresponding to the vector field ζ yields interesting information on the geometry of $\left(N^k, g\right)$. We could simultaneously impose some suitable restrictions on the operators T_ζ and Ψ_ζ, respectively, to facilitate this study.

Author Contributions: Conceptualization, N.B.T., S.D. and G.-E.V.; methodology, N.B.T., S.D. and G.-E.V.; software, N.B.T., S.D. and G.-E.V.; validation, N.B.T., S.D. and G.-E.V.; formal analysis, N.B.T., S.D. and G.-E.V.; investigation, N.B.T., S.D. and G.-E.V.; resources, N.B.T., S.D. and G.-E.V.; data curation, N.B.T., S.D. and G.-E.V.; writing—original draft preparation, N.B.T., S.D. and G.-E.V.; writing—review and editing, N.B.T., S.D. and G.-E.V.; visualization, N.B.T., S.D. and G.-E.V.; supervision, N.B.T., S.D. and G.-E.V.; project administration, N.B.T., S.D. and G.-E.V.; funding acquisition, N.B.T. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to extend their sincere appreciations to the supporting project, number (RSP2023R413), King Saud University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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