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On the aspects of enriched lattice-valued topological groups and closure of lattice-valued subgroups

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Dedicated to Professor John N. Mordeson on the occasion of his 87th birthday

Abstract. Starting with $\mathbb{L}$ as an enriched *cl*-premonoid, in this paper, we explore some categorical connections between $\mathbb{L}$-valued topological groups and Kent convergence groups, where it is shown that every $\mathbb{L}$-valued topological group determines a well-known Kent convergence group, and conversely, every Kent convergence group induces an $\mathbb{L}$-valued topological group. Considering an $\mathbb{L}$-valued subgroup of a group, we show that the category of $\mathbb{L}$-valued groups, $\mathbb{L}$-**GRP** has initial structure. Furthermore, we consider a category $\mathbb{L}$-**CLS** of $\mathbb{L}$-valued closure spaces, obtaining its relation with $\mathbb{L}$-valued Moore closure, and provide examples in relation to $\mathbb{L}$-valued subgroups that produce Moore collection. Here we look at a category of $\mathbb{L}$-valued closure groups, $\mathbb{L}$-**CLGRP** proving that it is a topological category. Finally, we obtain a relationship between $\mathbb{L}$-**GRP** and $\mathbb{L}$-**TransTOLGRP**, the category of $\mathbb{L}$-transitive tolerance groups besides adding some properties of $\mathbb{L}$-valued closures of $\mathbb{L}$-valued subgroups on $\mathbb{L}$-valued topological groups.

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1. Introduction

We have investigated a notion of $\mathbb{L}$-valued topological groups in [3], where we considered $\mathbb{L}$-valued subgroup of a group. Various aspects of $\mathbb{L}$-valued subgroups of groups are studied over the years by various authors, cf. [11, 23, 25, 26, 29] but its categorical behaviors are explored in a certain extent in recent times [26], although the category of fuzzy sets being studied for quite a long time, cf. [14, 33]. In [3], we also considered $\mathbb{L}$-valued closure of an $\mathbb{L}$-valued subgroup of a group in the context of $\mathbb{L}$-valued neighborhood groups, where the lattice under consideration was a complete MV-algebra with square roots.
Although our main objective of this paper is to explore further $\mathbb{L}$-valued subgroups from categorical view point and study category of $\mathbb{L}$-valued closure spaces vis-à-vis category of

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$\mathbb{L}$-valued closures groups in conjunction with $\mathbb{L}$-valued topological groups, we add some results on the connection of $\mathbb{L}$-valued topological groups and classical Kent convergence groups. However, we mainly focused on the impact of $\mathbb{L}$-valued closure structures on $\mathbb{L}$-valued topological groups instead of convergence groups. We arrange our work as follows. In Section 2, we give a short survey on $\mathbb{L}$-valued structures that we used in the text. The idea of convergence spaces and their connection to topological spaces is quite old, cf. [4–7, 10, 13, 20, 21, 27, 28]; following the concept of the compatibility of convergence structures with groups structures as proposed by D. C. Kent [20], for the first time, we explore a connection between the categories of $\mathbb{L}$-valued topological groups and Kent convergence groups, this is done in Section 3. We introduce the concept of $\mathbb{L}$-valued closure space, and $\mathbb{L}$-closure of $\mathbb{L}$-valued subgroup of a group in Section 4; we also introduce here a category of $\mathbb{L}$-valued closure groups - a topological category. With the help of connections, as presented by L. N. Stout in [32] and C. L. Waker in [33] between the categories of $\mathbb{L}$-**SET** and $\mathbb{L}$-**TOL**, the category of $\mathbb{L}$-valued tolerance spaces [32], we prove a connection between $\mathbb{L}$-**GRP**, category of $\mathbb{L}$-valued subgroups, and $\mathbb{L}$-valued transitive tolerance spaces, $\mathbb{L}$-**TranTOL**. Section 5 is devoted to study properties of $\mathbb{L}$-valued closure of $\mathbb{L}$-valued subgroups in the context of $\mathbb{L}$-valued topological groups, where some properties from groups are taken into consideration.

2. Preliminaries

Throughout the text we consider $\mathbb{L} = (\mathbb{L}, \leq)$ a complete lattice with $\top$, the top element and $\bot$, the bottom element of $\mathbb{L}$.

Definition 1. *[16, 17] A triple $(\mathbb{L}, \leq, *)$, where $*\colon \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$ is a binary operation on $\mathbb{L}$, is called a **G**$\mathbb{L}$-monoid if and only if the following holds:*
(GLM1) $(\mathbb{L}, *)$ is a commutative semigroup;
(GLM2) $\forall \alpha \in \mathbb{L}$: $\alpha * \top = \alpha$,
(GLM3) $*$ is distributive over arbitrary joins:
$\gamma * \left(\bigvee_{k\in K} \alpha_k\right) = \bigvee_{k\in K}(\gamma * \alpha_k)$, for $k \in K$, $\alpha_k, \gamma \in \mathbb{L}$;
(GLM4) for every $\gamma \leq \alpha$ there exists $\beta \in \mathbb{L}$ such that $\gamma = \alpha * \beta$ (divisibility).

The triple $(\mathbb{L}, \leq, *)$ is called a *commutative quantale* if (GLM1)-(GLM3) are fulfilled. If $* = \wedge$, then the triple $(\mathbb{L}, \leq, \wedge)$ is called a frame or a complete Heyting algebra.
For a commutative quantale, the implication operator $\rightarrow$, also known as residuum, is given by

$$\rightarrow\colon \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L},\ \alpha \rightarrow \beta = \bigvee\{\gamma \in \mathbb{L} |\, \alpha * \gamma \leq \beta\}.$$

A **G**$\mathbb{L}$-monoid $(\mathbb{L}, \leq, *)$ is called a complete **MV**-algebra if

$$\forall \alpha \in \mathbb{L},\ (\alpha \rightarrow \bot) \rightarrow \bot = \alpha \text{ (double negation).}$$

This means, in particular, that the unary operation $\neg\colon \mathbb{L} \longrightarrow \mathbb{L}, \alpha \mapsto \neg\alpha = \alpha \rightarrow \bot$ is an order-reversing involution.

Definition 2. *[16, 17] A triple* $(\mathbb{L}, \leq, \otimes)$*, where* $\otimes\colon \mathbb{L}\times\mathbb{L} \longrightarrow \mathbb{L}$ *is a binary operation on* $\mathbb{L}$*, is called a co-premonoid if and only if the following conditions are fulfilled:*
(CP1) $\forall \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{L}$*:* $\alpha_1 \leq \beta_1$ *and* $\alpha_2 \leq \beta_2$ *implies* $\alpha_1 \otimes \alpha_2 \leq \beta_1 \otimes \beta_2$*;*
(CP2) $\forall \alpha \in \mathbb{L}$*:* $\alpha \leq \alpha \otimes \top$ *and* $\alpha \leq \top \otimes \alpha$*.*

The category **COPML** consists of all co-premonoids as objects and morphisms as the mappings $\iota\colon (\mathbb{L}_1, \leq_1, \otimes_1) \longrightarrow (\mathbb{L}_2, \leq_2, \otimes_2)$ satisfying the following conditions:
(CPM1) ι preserves arbitrary joins;
(CPM2) $\iota(\alpha \otimes_1 \alpha') = \iota(\alpha) \otimes_2 \iota(\alpha')$, $\forall \alpha, \alpha' \in \mathbb{L}_1$;
(CPM3) ι preserves universal upper bounds; i.e., $\iota(\top) = \top$.

Definition 3. *[16, 17] A co-premonoid* $(\mathbb{L}, \leq, \otimes)$ *is called a cl-premonoid if and only if* *(CP3)* $\gamma \otimes \left(\bigvee_{k\in K} \alpha_k\right) = \bigvee_{k\in K}(\gamma \otimes \alpha_k)$*, and* $\left(\bigvee_{k\in K} \alpha_k\right) \otimes \gamma = \bigvee_{k\in K}(\alpha_k \otimes \gamma)$ *for* $K \neq \emptyset$*,* $k \in K$*,* $\alpha_k, \gamma \in \mathbb{L}$*, is satisfied.*

Definition 4. *[16, 17] The quadruple* $(\mathbb{L}, \leq, *, \otimes)$ *is called an* enriched *cl*-premonoid *if and only if the following are fulfilled: (CLP1)* $(\mathbb{L}, \leq, *)$ *is a* ***G***$\mathbb{L}$*-monoid;*
(CLP2) $(\mathbb{L}, \leq, \otimes)$ *is a cl-premonoid;*
(CLP3) $*$ *is dominated by* $\otimes$*:* $\forall \alpha, \beta, \gamma, \delta \in \mathbb{L}$*,*

$$(\alpha \otimes \beta) * (\gamma \otimes \delta) \leq (\alpha * \gamma) \otimes (\beta * \delta).$$

Definition 5. *[16, 17] A* ***G***$\mathbb{L}$*-monoid* $(\mathbb{L}, \leq, *)$ *is said to have square roots if and only if there exists a unary operator* $\mathcal{S}\colon \mathbb{L} \longrightarrow \mathbb{L}$ *such that the conditions below are satisfied:*
(S1) $\mathcal{S}(\alpha) * \mathcal{S}(\alpha) = \alpha$*,* $\forall \alpha \in \mathbb{L}$*;*
(S2) $\beta * \beta \leq \alpha$ *implies* $\beta \leq \mathcal{S}(\alpha)$*.*
Since the formation of square roots is uniquely determined by (S1) and (S2), $\mathcal{S}(\alpha)$ *is also written as* $\alpha^{\frac{1}{2}}$*.*
A ***G***$\mathbb{L}$*-monoid with square roots satisfies (S3) if it fulfills the following axiom:*
(S3) $(\alpha * \beta)^{\frac{1}{2}} = \left(\alpha^{\frac{1}{2}} * \beta^{\frac{1}{2}}\right) \vee \bot^{\frac{1}{2}}$, $\forall \alpha, \beta \in \mathbb{L}$.

If $\mathbb{L} = (\mathbb{L}, \leq, *)$ is a **G**$\mathbb{L}$-monoid with square roots, then the monoidal mean operator $\circledast\colon \mathbb{L}\times\mathbb{L} \longrightarrow \mathbb{L}$ is given by

$$\alpha \circledast \beta = \alpha^{\frac{1}{2}} * \beta^{\frac{1}{2}}, \forall \alpha, \beta \in \mathbb{L}.$$

An enriched *cl*-premonoid $\mathbb{L} = (\mathbb{L}, \leq, *, \otimes)$ is said to be *pseudo-bisymmetric* if it satisfies the following axiom:

$$(\alpha * \beta) \otimes (\gamma * \delta) = ((\alpha \otimes \gamma) * (\beta \otimes \delta)) \bigvee ((\alpha \otimes \bot) * (\beta \otimes \top)) \bigvee ((\bot \otimes \gamma) * (\top \otimes \delta)),$$
$$\forall \alpha, \beta, \gamma, \delta \in \mathbb{L}.$$

Remark 1. *[16, 17] (1) If* $(L, \leq, *)$ *is a* ***G****L-monoid with square roots, satisfying (S3), and* $\otimes$ *is the monoidal mean operator* $\circledast$*, then the quadruple* $(\mathbb{L}, \leq, *, \otimes)$ *is pseudo-bisymmetric.*
(2) If the cl-premonoid operation $\otimes$ *is identical to the quantal operation* $*$*, that is,* $\otimes = *$*, then the triple* $(\mathbb{L}, \leq, *, \otimes)$ *is pseudo-bisymmetric.*

Proposition 1. *[18] Let* $(\mathbb{L}, \leq, *)$ *be a GL-monoid. Then the following are fulfilled* $\forall \alpha, \beta, \gamma, \delta, \alpha_j, \beta_j, \gamma_j \in \mathbb{L}$*:*
(1) $\alpha \leq \beta \to \gamma \Leftrightarrow \alpha * \beta \leq \gamma$*;*
(2) $\alpha * (\alpha \to \beta) \leq \beta$;
(3) $\alpha \leq \beta \Rightarrow \alpha \to \gamma \leq \beta \to \gamma$;
(4) $\alpha \leq \beta \Rightarrow \gamma \to \alpha \geq \gamma \to \beta$*;*
(5) $(\alpha \to \beta) \to \beta \geq \alpha$*;*
(6) $\alpha * (\beta \to \gamma) \leq \beta \to (\alpha * \gamma)$*;*
(7) $\alpha \to (\bigwedge_{j \in J} \beta_j) = \bigwedge_{j \in J} (\alpha \to \beta_j)$*;*
(8) $(\bigvee_{j \in J} \alpha_j) \to \beta = \bigwedge_{j \in J} (\alpha_j \to \beta)$*;*
(9) if $\alpha, \beta \in \mathbb{L}$ *with* $\alpha \leq \beta$*, then for any* $\gamma \in \mathbb{L}$*,* $\gamma * \alpha \leq \gamma * \beta$*;*
(10) $\bigwedge_{j \in J} (\alpha_j * \gamma_j) \geq \left(\bigwedge_{j \in J} \alpha_j\right) * \left(\bigwedge_{j \in J} \gamma_j\right)$*;*
(11) $(\alpha \to \gamma) * (\beta \to \delta) \leq \alpha * \beta \to \gamma * \delta$;
(12) $\alpha \leq \beta \Leftrightarrow \alpha \to \beta = \top$*;*
(13) $\alpha \to \top = \top$*,* $\top \to \alpha = \alpha$*, and* $\bot \to \alpha = \top$*.*

In what follows, the quadruple $\mathbb{L} = (\mathbb{L}, \leq, *, \otimes)$ (or simply $\mathbb{L}$) is assumed to be an enriched *cl*-premonoid, where $*$ is reserved for the **G***L*-monoid operation, $\otimes$ is for *cl*-premonoid, unless otherwise specified. The set of all $\mathbb{L}$-sets or $\mathbb{L}$-valued sets and is denoted by $\mathbb{L}^X (= \{\nu \colon X \longrightarrow \mathbb{L}\})$. If $f \colon X \to Y$ is a function, then $f^{\leftarrow} \colon \mathbb{L}^Y \longrightarrow \mathbb{L}^X$ is defined for any $\mu \in \mathbb{L}^Y$ by $f^{\leftarrow}(\mu) = \mu \circ f$; and $f^{\rightarrow} \colon \mathbb{L}^X \longrightarrow \mathbb{L}^Y$ is defined by

$$f^{\rightarrow}(\nu)(y) = \bigvee \{\nu(x) | f(x) = y\},$$

for all $\nu \in \mathbb{L}^X, y \in Y$.
If $\cdot$ is a binary operation on a set X, then we define the binary operation $\odot$ on $\mathbb{L}^X$ as follows. For $\nu_1, \nu_2 \in \mathbb{L}^X$ and $z \in X$

$$\nu_1 \odot \nu_2(z) = \bigvee \{\nu_1(x) * \nu_2(y) |\, x, y \in X, x \cdot y = z\};$$

usually, we write xy instead of $x \cdot y$. If $\nu_1, \nu_2 \in \mathbb{L}^X$, and $\to$, $*$, $\otimes$ are operations on $\mathbb{L}$ as explained before, then these operations are carried over to $\mathbb{L}^X$ point-wise:
(i) $(\nu_1 \to \nu_2)(x) = \nu_1(x) \to \nu_2(x)$;
(ii) $(\nu_1 * \nu_2)(x) = \nu_1(x) * \nu_2(x)$;
(iii) $(\nu_1 \otimes \nu_2)(x) = \nu_1(x) \otimes \nu_2(x)$, $\forall x \in X$.

Definition 6. *[17, 18] A map* $\mathcal{F} \colon \mathbb{L}^X \longrightarrow \mathbb{L}$ *is called an* $\mathbb{L}$-valued filter *on* X *if and only if the conditions below are satisfied:*
(LF1) $\mathcal{F}(\top_X) = \top$*,* $\mathcal{F}(\bot_X) = \bot$ *;*
(LF2) if $\nu_1, \nu_2 \in \mathbb{L}^X$ *with* $\nu_1 \leq \nu_2$*, then* $\mathcal{F}(\nu_1) \leq \mathcal{F}(\nu_2)$*;*
(LF3) $\mathcal{F}(\nu_1) \otimes \mathcal{F}(\nu_2) \leq \mathcal{F}(\nu_1 \otimes \nu_2)$, $\forall \nu_1, \nu_2 \in \mathbb{L}^X$*.*
(S$\mathbb{L}$*) An* $\mathbb{L}$*-valued filter* $\mathcal{F}$ *is called a* stratified $\mathbb{L}$-valued filter *if* $\forall \alpha \in \mathbb{L}, \forall \mu \in \mathbb{L}^X$, $\alpha * \mathcal{F}(\mu) \leq \mathcal{F}(\alpha * \mu)$*.*

The set of all stratified $\mathbb{L}$-valued filters on X is denoted by $\mathcal{F}^s_{\mathbb{L}}(X)$. On $\mathcal{F}^s_{\mathbb{L}}(X)$, partial ordering $\leq$ is defined by: if $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{L}}(X)$, then $\mathcal{F} \leq \mathcal{G} \Leftrightarrow \mathcal{F}(\nu) \leq \mathcal{G}(\nu)$, $\forall \nu \in \mathbb{L}^X$. If

$x \in X$, then $[x] \in \mathcal{F}^s_{\mathbb{L}}(X)$, called point stratified $\mathbb{L}$-valued filter on X, and is defined as $[x](\nu) = \nu(x)$, for all $\nu \in \mathbb{L}^X$.
If $\mathcal{F} \in \mathcal{F}^s_{\mathbb{L}}(X)$, then the stratified $\mathbb{L}$-valued filter $f^{\Rightarrow}(\mathcal{F})\colon \mathbb{L}^Y \to \mathbb{L}$ on Y is defined for any $\mu \in \mathbb{L}^Y$ by

$$[f^{\Rightarrow}(\mathcal{F})](\mu) = \mathcal{F}(f^{\leftarrow}(\mu)) = \mathcal{F}(\mu \circ f).$$

If $\mathcal{F} \in \mathcal{F}^s_{\mathbb{L}}(Y)$, then $f^{\Leftarrow}(\mathcal{F})\colon \mathbb{L}^X \to \mathbb{L}$ is defined by

$$[f^{\Leftarrow}(\mathcal{F})](\nu) = \bigvee\{\mathcal{F}(\mu) | \mu \in \mathbb{L}^Y, f^{\leftarrow}(\mu) \le \nu\},$$

for all $\nu \in \mathbb{L}^X$, is a stratified $\mathbb{L}$-filter on X if and only if for all $\mu \in \mathbb{L}^Y, f^{\leftarrow}(\mu) = \perp_X \Rightarrow \mathcal{F}(\mu) = \perp$.
If $\nu \in \mathbb{L}^X$ and $\mu \in \mathbb{L}^Y$, then the product $\nu \times \mu\colon X \times Y \longrightarrow \mathbb{L}$ is defined by:

$$\nu \times \mu = \nu \circ pr_1 * \mu \circ pr_2,$$

where $pr_1 : X \times Y \to X, (x, y) \mapsto x$ and $pr_2 : X \times Y \to Y, (x, y) \mapsto y$ are usual projections. Note that in the preceding definition of product $\mathbb{L}$-set the operation $*$ holds only for finite case; otherwise, we need to take $* = \wedge$.

Proposition 2. *[16] If $(L, \le, *)$ is a **G**L-monoid, then for stratified L-valued filters $\mathcal{F}_1$ and $\mathcal{F}_2$, the supremum $\mathcal{F}_1 \vee \mathcal{F}_2$ exists if and only if $\mathcal{F}_1(\nu_1) * \mathcal{F}_2(\nu_2) = \perp\ \forall \nu_1, \nu_2 \in L^X$ such that $\nu_1 * \nu_2 = \perp_X$. In particular, the supremum is the stratified L-valued filter defined for all $\nu \in L^X$ by*

$$\mathcal{F}_1 \vee \mathcal{F}_2(\nu) = \bigvee\{\mathcal{F}_1(\nu_1) * \mathcal{F}_2(\nu_2) |\, \nu_1, \nu_2 \in L^X, \nu_1 * \nu_2 \le \nu\}.$$

Let $(G, \cdot)$ be a group. If $\mathcal{F} \in \mathbb{L}^s(G)$, then $\mathcal{F}^{-1}$ is defined by $\mathcal{F}^{-1}(\nu) = \mathcal{F}(\nu^{-1})$, where $\nu^{-1}\colon G \longrightarrow \mathbb{L}, x \longmapsto \nu(x^{-1})$. Clearly, $\mathcal{F}^{-1} \in \mathcal{F}^s_{\mathbb{L}}(G)$, since for any $\nu \in L^X$, $\jmath^{\Rightarrow}(\mathcal{F})(\nu) = \mathcal{F}(\jmath^{\leftarrow}(\nu)) = \mathcal{F}(\nu^{-1}) = \mathcal{F}^{-1}(\nu)$, where $\jmath\colon G \longrightarrow G, x \mapsto x^{-1}$. Also, if $m\colon G \times G \to G, (g, h) \mapsto gh$, then for any $\nu_1, \nu_2 \in \mathbb{L}^G$ and $z \in G$, $m^{\rightarrow}(\nu_1 \times \nu_2)(z) = \bigvee_{m(g,h)=z} (\nu_1 \times \nu_2)(g, h)$ $= \bigvee_{gh=z} (\nu_1 \circ pr_1 * \nu_2 \circ pr_2)(g, h) = \bigvee_{gh=z} \nu_1 \circ pr_1(g, h) * \nu_2 \circ pr_2(g, h) = \bigvee_{gh=z} \nu_1(g) * \nu_2(h) = \nu_1 \odot \nu_2(z)$.

Lemma 1. *[3] Let $\mathbb{L} = (\mathbb{L}, \le, *)$ be a **G**L-monoid and $(G, \cdot) \in |\boldsymbol{GRP}|$. Then for any $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{L}}(X)$, $m^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) = \mathcal{F} \odot \mathcal{G}$.*

Definition 7. *[17] Consider a mapping $\mathfrak{N}\colon X \longrightarrow \mathbb{L}^{\mathbb{L}^X}$ such that the following conditions are fulfilled:*
(LN1) $\mathfrak{N}^x(\top_X) = \top$;
(LN2) $\mathfrak{N}^x(\nu_1) \le \mathfrak{N}^x(\nu_2)$ for all $\nu_1, \nu_2 \in \mathbb{L}^X$ with $\nu_1 \le \nu_2$;
(LN3) $\mathfrak{N}^x(\nu_1) \otimes \mathfrak{N}^x(\nu_2) \le \mathfrak{N}^x(\nu_1 \otimes \nu_2)$, for all $\nu_1, \nu_2 \in \mathbb{L}^X$;
(LN4) $\mathfrak{N}^x(\nu) \le \nu(x)$, for all $\nu \in \mathbb{L}^X$;
(LN5) $\forall x \in X$ and $\nu \in \mathbb{L}^X$, $\mathfrak{N}^x(\nu) \le \bigvee\{\mathfrak{N}^x(\mu)\colon\ \mu \in \mathbb{L}^X, \mu(y) \le [\mathfrak{N}^y](\nu), \forall y \in X\}$
(SLN) $\alpha * \mathfrak{N}^x(\nu) \le \mathfrak{N}^x(\alpha * \nu)$.
Then $\mathfrak{N} = (\mathfrak{N}^x)_{x \in X}$ is called a stratified $\mathbb{L}$-valued neighborhood system *on X, and the*

pair $(X, \mathfrak{N} = (\mathfrak{N}^x)_{x\in X})$ *is called a* stratified $\mathbb{L}$-valued neighborhood space.
If $(X, \mathfrak{N})$ *and* $(Y, \mathfrak{M})$ *stratified* $\mathbb{L}$*-valued neighborhood spaces, then a map* $f\colon (X, \mathfrak{N}) \to (Y, \mathfrak{M})$ *is said to be* continuous at a point $x \in X$ *if and only if* $\mathfrak{M}^{f(x)}(\nu) \leq \mathfrak{N}^x(f^{\leftarrow}(\nu))$*, for all* $\nu \in \mathbb{L}^Y$.
S$\mathbb{L}$-***NS*** *denotes the category of all stratified* $\mathbb{L}$*-valued neighborhood spaces as objects and all continuous maps as morphisms.*

Definition 8. *[17, 22] Let* $\Delta \subseteq \mathbb{L}^X$ *such that the following are fulfilled:*
(LT1) $\top_X, \bot_X \in \Delta$;
(LT2) $\nu_1, \nu_2 \in \Delta \Rightarrow \nu_1 \otimes \nu_2 \in \Delta$;
(LT3) $\{\nu_j\}_{j\in J} \subseteq \Delta \Rightarrow \bigvee_{j\in J} \nu_j \in \Delta$;
(SLT) $\nu \in \Delta$, $\alpha \in \mathbb{L} \Rightarrow \alpha_X * \nu \in \Delta$.
We call Δ *an* $\mathbb{L}$*-valued topology on* X *if it satisfies (LT1)-(LT3), and the pair* (X, Δ) *is called an* $\mathbb{L}$*-valued topological space. If* Δ *satisfies (LT1)-(SLT) then we call it a stratified* $\mathbb{L}$*-valued topology on* X *and the pair* (X, Δ) *or* X *in short, if there is no confusion, is called a* stratified L-valued topological space*; members of* Δ *are called* open $\mathbb{L}$-valued sets or $\mathbb{L}$-valued subsets*; the members of* $\Theta(X) = \{\xi \in \mathbb{L}^X : \xi^c \text{ is open}\}$ *are called* closed $\mathbb{L}$-valued sets or $\mathbb{L}$-valued subsets*, where* ξ^c *is the so-called* qusi-complementation *of* ξ*. Note that* $\Theta(X)$ *is closed under formation of arbitrary* inf*s and finite* sup*s. Furthermore, recall that the closure of* $\nu \in \mathbb{L}^X$*, denoted by* $\overline{\nu}^X$ *is defined as:* $\overline{\nu}^X = \bigwedge\{\theta \in \Theta(X) \colon \nu \leq \theta\}$.
If (X, Δ) *and* (Y, Γ) *are stratified* $\mathbb{L}$*-valued topological spaces, then a function* $f\colon (X, \Delta) \to (Y, \Gamma)$ *is said to be* continuous *if and only if for any* $\sigma \in \Gamma$, $f^{\leftarrow}(\sigma) \in \Delta$*. The category* ***S***$\mathbb{L}$-***TOP*** *consists of all stratified* $\mathbb{L}$*-valued topological spaces as objects and all continuous maps between them as morphisms, while the category* $\mathbb{L}$-***TOP*** *consisting of all* $\mathbb{L}$*-valued topological spaces as objects and all continuous maps between them as morphisms.*

Every stratified $\mathbb{L}$-valued topology Δ on X induces a stratified $\mathbb{L}$-valued neighborhood system $\mathfrak{N}_\Delta = (\mathfrak{N}^x_\Delta)$ as follows:

$$\mathfrak{N}^x_\Delta(\mu) = \bigvee\{\nu(x) \colon \nu \in \Delta,\ \nu \leq \mu\}, \text{ for all } \mu \in \mathbb{L}^X \text{ and } x \in X.$$

Conversely, every stratified $\mathbb{L}$-valued neighborhood system $\mathfrak{N} = (\mathfrak{N}^x)_{x\in X}$ on X induces a stratified $\mathbb{L}$-valued topology $\Delta_{\mathfrak{N}}$ on X :

$$\Delta_{\mathfrak{N}} = \{\nu \in \mathbb{L}^X : \nu(x) \leq \mathfrak{N}^x(\nu),\ \forall x \in X\}.$$

It follows that the interrelationship between $\mathbb{L}$-valued neighborhood system and $\mathbb{L}$-valued topologies can be viewed as:

$$\nu \in \Delta \Leftrightarrow \nu(x) \leq \mathfrak{N}^x(\nu),\ \forall x \in X \qquad (\dagger).$$

As a consequence of (†) it follows that the continuity between the objects in **S**$\mathbb{L}$**-TOP**, and the continuity between objects in **S**$\mathbb{L}$**-NS** are equivalent concept, cf. [18].

3. $\mathbb{L}$-valued topological groups and Kent convergence groups

We consider $\mathbb{L} = (\mathbb{L}, \leq, *, \otimes = *)$ an enriched *cl*-premonoid, where $*$ is a G$\mathbb{L}$-monoid operation. Let the category of groups and group homomorphisms be denoted by **GRP**.

Definition 9. *Let* $(X, \cdot) \in |\boldsymbol{GRP}|$ *and* $(X, \Delta) \in |\boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{TOP}|$. *Then the triple* $(X, \cdot, \Delta)$ *is called* a stratified $\mathbb{L}$-valued topological group *if and only if the conditions below are fulfilled:*
(LTGM) the mapping $m\colon (X \times X, \Delta \times \Delta) \longrightarrow (X, \Delta)$, $(x, y) \longmapsto xy$ *is continuous ;*
(LTGI) the mapping $\jmath\colon (X, \Delta) \longrightarrow (X, \Delta)$, $x \longmapsto x^{-1}$ *is continuous.*
The category of all stratified $\mathbb{L}$*-valued topological groups and continuous group homomorphisms is denoted by* $\boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{TOPGRP}$.

Definition 10. *[3] Let* $(X, \cdot) \in |\boldsymbol{GRP}|$ *and* $\left(X, \mathfrak{N} = (\mathfrak{N}^x)_{x \in X}\right) \in |\boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{NS}|$.
Then the triple $\left(X, \cdot, \mathfrak{N} = (\mathfrak{N}^x)_{x \in X}\right)$ *is called* a stratified $\mathbb{L}$-valued neighborhood group *if and only if*
(LNGM) $\mathfrak{N}^{xy} \leq \mathfrak{N}^x \odot \mathfrak{N}^y$, *and (LNGI)* $\mathfrak{N}^{x^{-1}} \leq (\mathfrak{N}^x)^{-1}$ *are satisfied, where for any* $\xi \in \mathbb{L}^G$*:*
$\mathfrak{N}^x \odot \mathfrak{N}^y(\xi) = m^{\Rightarrow}(\mathfrak{N}^x \times \mathfrak{N}^y)(\xi) = \bigvee\{\mathfrak{N}^x(\xi_1) \wedge \mathfrak{N}^y(\xi_2)\colon \xi_1, \xi_2 \in \mathbb{L}^X, \xi_1 \times \xi_2 \leq m^{\leftarrow}(\xi)\}$.
A stratified $\mathbb{L}$*-valued neighborhood system on a group* X *is said to be compatible with the group structure of* X *if and only if the group operations are continuous; i.e., conditions (LNGM) and (LNTGI) are fulfilled.*
The category $\boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{NS}$ *consists of all stratified* $\mathbb{L}$*-valued neighborhood groups as objects and continuous group homomorphisms as morphisms.*

Example 1. *Let* $(G, \cdot) \in |\boldsymbol{GRP}|$, *and* $\mathfrak{N}^i\colon \mathbb{L}^X \longrightarrow \mathbb{L}$ *defined by* $\mathfrak{N}^i = \bigwedge_{x \in G}[x]$. *Then the triple* $(G, \cdot, \mathfrak{N}^i)$ *is a stratified* $\mathbb{L}$*-valued neighborhood group, called* indiscrete stratified $\mathbb{L}$-valued neighborhood group.

Example 2. *Let* $(G, \cdot) \in |\boldsymbol{GRP}|$, *and* $\mathfrak{N}^d\colon \mathbb{L}^X \longrightarrow \mathbb{L}$ *defined by* $\mathfrak{N}^{xd}(\nu) = \nu(x)$. *Then the triple* $(G, \cdot, \mathfrak{N}^d)$ *is a stratified* $\mathbb{L}$*-valued neighborhood group, called* discrete stratified $\mathbb{L}$-valued neighborhood group.

Lemma 2. *[3] Let* $(G, \cdot, \Delta) \in |\boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{TOPGRP}|$, *and* $a \in G$. *Then the translations (left and right)* $\mathcal{L}_a\colon (G, \cdot, \Delta) \longrightarrow (G, \cdot, \Delta), g \longmapsto ag$, *and* $\mathcal{L}_x\colon (G, \cdot, \Delta) \longrightarrow (G, \cdot, \Delta), g \longmapsto ga$ *are homeomorphisms. Also the mapping* $\mathcal{C}_a\colon (G, \cdot, \Delta) \longrightarrow (G, \cdot, \Delta), g \longmapsto gag^{-1}$ *the inner automorphism is an isomorphism.*

Definition 11. *[20, 27] A* Kent convergence structure q *on* X *is a subset* $q \subseteq \mathbb{F}(X) \times X$ *such that the following conditions are satisfied:*
(C1) $x \in q(\dot{x}), \forall x \in X$, *where* $\dot{x}$ *denotes the ordinary principal filter on* X *generated by the singleton* $\{x\}$*;*
(C2) $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, $\mathbb{F} \subseteq \mathbb{G}$, $x \in q(\mathbb{F})$ *implies* $x \in q(\mathbb{G})$*;*
(C3) $x \in q(\mathbb{F})$ *implies* $x \in q(\mathbb{F} \cap \dot{x})$.
Note that in [4], [6] and [7] the above notion is called a local filter convergence structure q *on* X*, however.*
A mapping $f\colon (X, q) \longrightarrow (X', q')$ *is called* continuous *if for all* $\mathbb{F} \in \mathbb{F}(X)$ *and* $x \in X$, $x \in$

*$q(\mathbb{F})$ implies $f(x) \in q(f(\mathbb{F}))$. The category of all Kent convergence spaces and continuous mapping is denoted by **KCONV**. The category **KCONV** is a strong topological universe, cf. [10, 28].*
The pair (X, q) is called a limit space *if conditions (C1), (C2) and (C4): $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, $x \in q(\mathbb{F})$ and $x \in q(\mathbb{G})$ implies $x \in q(\mathbb{F} \cap \mathbb{G})$.*
*The category of limit spaces is denoted by **LIM**. A limit structure q on X is called* a principal limit structure *on X if and only if for every $x \in X$ there exists a unique filter $\mathbb{U}_x \in \mathbb{F}(X)$ such that the following relation holds:*

$$q = \{(\mathbb{F}, x) \in \mathbb{F}(X) \times X \colon \mathbb{U}_x \subseteq \mathbb{F}\}.$$

*The category of all principal limit spaces and continuous mappings is denoted by **pLIM**.*

Remark 2. *It is important to mention here that the categories of closure spaces, **CLS**, and **LIM** with principal limit structures are isomorphic, cf. [28], we are not interested at this stage to carry out research in this direction, and postpone it for further investigation.*

Definition 12. *[27] Let $(G, \cdot) \in |\boldsymbol{GRP}|$ and $(G, q) \in |\boldsymbol{KCONV}|$ (resp. $(G, q) \in |\boldsymbol{LIM}|$). Then the triple $(G, \cdot, q) \in |\boldsymbol{KCONVGRP}|$ (resp. $(G, \cdot, q) \in |\boldsymbol{LIMGRP}|$) if the following are fulfilled:*
(CGM) $x \in q(\mathbb{F})$ and $y \in q(\mathbb{G})$ implies $xy \in q(\mathbb{F} \odot \mathbb{G})$;
(CGI) $x \in q(\mathbb{F})$ implies $x^{-1} \in q(\mathbb{F}^{-1})$.
*The category of all Kent convergence groups and group homomorphisms is denoted by **KCONVGRP** (resp. the category of all limit groups and group homomorphisms is denoted by **LIMGRP**).*

Given a stratified L-topological space $(X, \Delta_{\mathfrak{N}})$ with the corresponding $\mathbb{L}$-neighborhood system $\mathfrak{N}$. Then a filter $\mathbb{F}$ is said to be *convergent to a point $x \in X$* (we denoted it as $x \in q_{\Delta_{\mathfrak{N}}}(\mathbb{F})$) with respect to $\Delta_{\mathfrak{N}}$ if and only if for all $\nu \in \mathbb{L}^X$ the following holds:

$$\mathfrak{N}_x(\nu) \leq \bigvee_{F \in \mathbb{F}} \Big(\bigwedge_{y \in F} \nu(y) \Big).$$

Lemma 3. *Let $(G, \cdot, \Delta_{\mathfrak{N}}) \in |\boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{TOPGRP}|$, where Δ is a stratified L-valued topology on G and $\mathfrak{N}$ is a corresponding $\mathbb{L}$-valued neighborhood system. Then $(G, \cdot, q_{\Delta_{\mathfrak{N}}}) \in |\boldsymbol{KCONVGRP}|$.*

Proof. Let $(G, \cdot, \Delta_{\mathfrak{N}}) \in |\mathbf{S}\mathbb{L}\text{-}\mathbf{TOPGRP}|$. Then in view of the Lemma 5.4.1[18], we only need to Check the conditions (CGM) and (CGI).
(CGM) Let for $\mathbb{F}, \mathbb{G} \in \mathbb{F}(G)$ and $x, y \in G$, $x \in q_{\Delta_{\mathfrak{N}}}(\mathbb{F})$ and $y \in q_{\Delta_{\mathfrak{N}}}(\mathbb{G})$. Then for any $\nu, \mu \in \mathbb{L}^G$: $\mathfrak{N}_x(\nu) \leq \bigvee_{F \in \mathbb{F}} \bigwedge_{y_1 \in F} \nu(y_1)$, and $\mathfrak{N}_y(\mu) \leq \bigvee_{G \in \mathbb{G}} \bigwedge_{y_2 \in G} \mu(y_2)$. Thus, for any $\sigma \in \mathbb{L}^G$,

$$\mathfrak{N}_{xy}(\sigma) \leq \bigvee \{\mathfrak{N}_x(\nu) * \mathfrak{N}_y(\mu) \colon \nu(x) * \mu(y) \leq \sigma(xy)\} \leq$$
$$\bigvee_{\nu(x) * \mu(y) \leq \sigma(xy)} \bigvee_{F \cdot G \in \mathbb{F} \odot \mathbb{G}} \bigwedge_{y_1 \in F, y_2 \in G} \nu(x) * \mu(y) \leq \bigvee_{F \cdot G \in \mathbb{F} \odot \mathbb{G}} \bigwedge_{xy \in F \cdot G} \sigma(xy)$$

This implies that $\mathfrak{N}_{xy}(\sigma) \leq \bigvee_{F\cdot G\in \mathbb{F}\odot\mathbb{G}} \bigwedge_{z\in F\cdot G} \sigma(xy)$, i.e., $xy \in q_{\Delta_{\mathfrak{N}}}(\mathbb{F}\odot\mathbb{G})$.
(CGI) Let $\mathbb{F} \in \mathbb{F}(G)$, and $x \in X$. Then by invoking (†) in conjunction with the Lemma 5.4.1[18], if we consider $x \in q_{\Delta_{\mathfrak{N}}}(\mathbb{F})$, then for any $\nu \in \mathbb{L}^G$, we have $\mathfrak{N}_x(\nu) \leq \bigvee_{F\in\mathbb{F}} \left(\bigwedge_{y\in F} \nu(y)\right)$.
Now due to the continuity of $\jmath$, we have
$\mathfrak{N}_{x^{-1}}(\nu) \leq \mathfrak{N}_x(\nu^{-1}) \leq \bigvee_{F\in\mathbb{F}} \left(\bigwedge_{y\in F} \nu^{-1}(y)\right) = \bigvee_{F^{-1}\in\mathbb{F}^{-1}} \left(\bigwedge_{y^{-1}\in F^{-1}} \nu(y^{-1})\right)$.
That is, $\mathfrak{N}_{x^{-1}}(\nu) \leq \bigvee_{F^{-1}\in\mathbb{F}^{-1}} \left(\bigwedge_{y^{-1}\in F^{-1}} \nu(y^{-1})\right)$ implying $x^{-1} \in q_{\Delta_{\mathfrak{N}}}(\mathbb{F}^{-1})$.

Remark 3. *Referring to the pp. 175 [18], one can observe that given a Kent convergence structure q on X, then q induces a stratified $\mathbb{L}$-valued topology $\widehat{\Delta}_q$ in the following way:*

$$\widehat{\Delta}_q = \{\sigma \in \mathbb{L}^X : \sigma(x) \leq \bigvee_{A\in\mathbb{F}} \left(\bigwedge_{z\in A} \sigma(z)\right),\ \forall \mathbb{F} \in \mathbb{F}(X),\ x \in q(\mathbb{F})\}$$

*From Lemma 5.4.2[18], it follows that there is a functor $\mathfrak{G}\colon$ **KCONV** $\longrightarrow$ **S**$\mathbb{L}$-**TOP**, where $\mathfrak{G}(X,q) = (X, \widehat{\Delta}_q)$ and $\mathfrak{G}(f) = f$.*

Lemma 4. *Let $(G,\cdot,q) \in$ |**KCONVGRP**|. Then $(G,\cdot,\widehat{\Delta}_q) \in$ |**S**$\mathbb{L}$-**TOPGRP**|.*

Proof. Let $(G,\cdot,q) \in$ |**KCONVGRP**|. Note that the product $\mathbb{L}$-valued topology on $\widehat{\Delta}_q \times \widehat{\Delta}_q$ is the initial $\mathbb{L}$-valued topology with respect to the projects $pr_1\colon X\times X \longrightarrow X$,$(x,y) \longmapsto x$, and $pr_2\colon X\times X \longrightarrow X$,$(x,y) \longmapsto y$. Further note that $\widehat{\Delta}_q \times \widehat{\Delta}_q = \{(\nu^1\cdot pr_1) * (\nu^2 \cdot pr_2)\colon \nu^1,\nu^2 \in \widehat{\Delta}_q\}$ is a base for the product $\mathbb{L}$-topology on $X\times X$, where the $\mathbb{L}$-set can be given by: $\mu_0 := \bigvee_{i\in I}\left(\nu_i^1\cdot pr_1) * (\mu_i^2\cdot pr_2)\right)$, and $\nu_i^1, \mu_i^2 \in \widehat{\Delta}_q$. Thus, we have for any $\nu \in \widehat{\Delta}_q$ and $(x,y) \in X\times X$, and due to the property of $*$ in $\mathbb{L}$:
$\nu(xy) = m^{\leftarrow}(\nu)(x,y) = \bigvee_{i\in I}\left[(pr_1^{\leftarrow}(\nu_i^1)(x,y)) * (pr_2^{\leftarrow}(\mu_i^2)(x,y)))\right]$ $(\nu_i^1, \mu_i^2 \in \widehat{\Delta}_q)$.
$= \bigvee_{i\in I}\left[\nu_i^1(x) * \mu_i^2(y)\right]$, $(\nu_i^1, \mu_i^2 \in \widehat{\Delta}_q)$.
$\leq \bigvee_{i\in I}\left[\bigvee_{A\in\mathbb{F}}\left(\bigwedge_{z_1\in A}\nu_i^1(z_1)\right) * \bigvee_{B\in\mathbb{G}}\left(\bigwedge_{z_2\in B}\nu_i^2(z_2)\right)\right]$
$\leq \bigvee_{i\in I}\left[\bigvee_{A.\cdot B\in\mathbb{F}\odot\mathbb{G}}\bigwedge_{z_1z_2\in A\cdot B}\left(\nu_i^1(z_1) * \nu_i^2(z_2)\right)\right]$
$= \left[\bigvee_{A\cdot B\in\mathbb{F}\odot\mathbb{G}}\bigwedge_{z_1z_2\in A\cdot B}\nu(z_1z_2)\right]$
That is, $\nu(xy) \leq \left[\bigvee_{H\in\mathbb{F}\odot\mathbb{G}}\bigwedge_{z_1z_2\in H}\nu(z_1z_2)\right]$ and $xy \in q(\mathbb{F}\odot\mathbb{G})$ due to the condition (CGM) implying $m^{\leftarrow}(\nu) \in \widehat{\Delta}_q \times \widehat{\Delta}_q$. This proves condition (LTGM).
Now let $x \in q(\mathbb{F})$ for any $\mathbb{F} \in \mathbb{F}(G)$ and let $\nu \in \widehat{\Delta}_q$. Then we have
$\jmath^{\leftarrow}(\nu)(x) = \nu(\jmath(x)) \leq \bigvee_{A\in\mathbb{F}}\left(\bigwedge_{z_2\in\jmath(A)}\nu(z_2)\right) = \bigvee_{A^{-1}\in\mathbb{F}^{-1}}\left(\bigwedge_{z_1\in A^{-1}}\jmath^{\leftarrow}(\nu)(z_1)\right)$,
that is, $\jmath^{\leftarrow}(\nu)(x) \leq \bigvee_{A^{-1}\in\mathbb{F}^{-1}}\left(\bigwedge_{z_1\in A^{-1}}\jmath^{\leftarrow}(\nu)(z_1)\right)$; and $x^{-1} \in q(\mathbb{F}^{-1})$ because of the condition (CGI). These together imply that $\jmath^{\leftarrow}(\nu) \in \widehat{\Delta}_q$, this proves (LTGI).

Theorem 1. *The functor $\mathfrak{F}$:* **S**$\mathbb{L}$-**TOPGRP**$\longrightarrow$ **KCONVGRP** *as defined below*

$$\mathfrak{F} : \begin{cases} \boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{TOPGRP} & \longrightarrow & \boldsymbol{KCONVGRP} \\ (G,\cdot,\Delta_{\mathfrak{N}}) & \longmapsto & (G,\cdot,q_{\Delta_{\mathfrak{N}}}) \\ f & \longmapsto & f \end{cases}$$

has a left adjoint.

Proof. In view of Lemma 3 in conjunction with Lemma 5.4.1 [18], $\mathfrak{F}\colon$ **S𝕃-TOPGRP**$\longrightarrow$ **KCONVGRP** is a functor. Define $\mathfrak{G}\colon$ **KCONVGRP** $\longrightarrow$ **S𝕃-TOPGRP** by

$$\mathfrak{G} : \begin{cases} \textbf{KCONVGRP} & \longrightarrow & \textbf{S}\mathbb{L}\textbf{-TOPGRP} \\ (G,\cdot,q) & \longmapsto & \left(G,\cdot,\widehat{\Delta}_q\right) \\ f & \longmapsto & f \end{cases}$$

Then from Lemma 4 in conjunction with Lemma 5.4.2 [18] that $\mathfrak{G}$ is a functor since in both the cases the group homomorphism structures remain unchanged. That the functor $\mathfrak{G}$ is a left adjoint since in both the cases group homomorphism structures remain unchanged. That the functor $\mathfrak{G}$ is a left adjoint to $\mathfrak{F}$ is an immediate consequence of the Proposition 5.4.3 [18].

4. Enriched lattice-valued subgroup of a group and enriched lattice-valued neighborhood groups

Definition 13. *Let $\mathbb{L} = (\mathbb{L}, \leq, \wedge, *)$ be an enriched cl-premonoid, $(G, \cdot) \in |\boldsymbol{GRP}|$. Then an $\mathbb{L}$-set $\mu\colon G \longrightarrow \mathbb{L}$ is called an $\mathbb{L}$-valued subgroup of a group G if and only if the following conditions are fulfilled:*
(LG1) $\mu(e) = \top$;
(LG2) $\mu(g) * \mu(h) \leq \mu(gh)$, $\forall g, h \in G$;
(LG3) $\mu(g) \leq \mu(g^{-1})$.
Then the pair $(G, \cdot, \mu)$ is called an $\mathbb{L}$-valued subgroup space. *Let $(H, \cdot, \xi)$ be another $\mathbb{L}$-valued subgroup of a group H. Define a mapping between $\mathbb{L}$-valued subgroup spaces, $f\colon (G, \cdot, \mu) \longrightarrow (H, \cdot, \xi)$ such that*

$$\mu(g) \leq \xi(f(g)), \ \forall g \in G \quad (\ddagger)$$

*The category of all $\mathbb{L}$-valued subgroup spaces and all group homomorphisms satisfying ($\ddagger$) is denoted by $\mathbb{L}$-**GRP**. Sometime we denote the set of $\mathbb{L}$-valued subgroups of a group G by $\mathbb{L}(G)$.*

Example 3. *[3] Let $\mathbb{L} = ([0, 1], \leq, \wedge, *)$ be an enriched cl-premonoid, where $*$ is a t-norm on $[0, 1]$. Let G be the cyclic group C_n of order n ($n \geq 1$) with a as the generator; specifically, $C_n = \{e, a, a^2, ..., a^{n-1};\, a^n = e\}$ with respect to multiplication $\cdot$.*
Define $\mu\colon\ G \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = e; \\ \frac{1}{n}, & \text{otherwise.} \end{cases}$$

Then $(G, \cdot, \mu)$ is an enriched lattice-valued subgroup space. In fact, for (LG1) $\mu(e) = 1$ while (LG3) follows from the definition. For (LG2), consider $x, y \in G$ with $x \neq e$ and $y \neq e$, then $\mu(x) * \mu(y) = \frac{1}{n} * \frac{1}{n} \leq \frac{1}{n} * 1 = \frac{1}{n}$ implying $\mu(x) * \mu(y) \leq \mu(xy)$; other choices follow similarly. Hence μ is an enriched lattice-valued subgroup of the group G.

Remark 4. *In [33], C. L. Walker pointed out that for a category of fuzzy subsets* $\mathsf{F} = \boldsymbol{Set}(\mathsf{I})$ *where all objects are* (X, ν), $X \in |\boldsymbol{Set}|$, *with* $\nu\colon X \longrightarrow \mathsf{I}$ *- a mapping from* X *to the unit interval. The morphisms* F *are all mappings* $f\colon (X,\nu) \longrightarrow (Y,\mu)$ *satisfying* $\nu(x) \leq \mu(f(x))$. *Furthermore, note that in [14], J. Goguen, defined the category* $\boldsymbol{SET}(\mathbb{L})$ *having objects the pair* (X,ν), *where* $\nu\colon X \longrightarrow \mathbb{L}$, *and morphisms* $f\colon (X,\nu) \longrightarrow (Y,\mu)$ *such that* $\nu(x) \leq \mu(f(x))$ *holds. L. Stout [32] argued that this category* $\boldsymbol{SET}(\mathbb{L})$ *has initial structure and is cartesian closed. The initial structure is given as: for a family of mappings* $(f_j\colon X \longrightarrow (Y_j,\mu_j))_j$, $\nu(x) = \bigwedge_j \mu_j(f_j(x))$ *gives the initial structure on* X. *The cartesian closed structure is obtained as:* $(\mathcal{C}(X,Y),\nabla)$, *where* $\nabla(f) = \bigwedge_{x\in X}[\nu(x) \longrightarrow \mu(f(x))]$, *where for all* $(f\colon (X,\nu) \longrightarrow (Y,\mu)) \in \mathcal{C}(X,Y)$, *and the implication* $\rightarrow$ *is given by:* $\nu(x) \longrightarrow \mu(f(x)) = \bigvee\{\lambda\colon \lambda \wedge \nu(x) \leq \mu(f(x))\}$.

Lemma 5. $\mathbb{L}$-$\boldsymbol{GRP}$ *has initial structure where the underlying forgetful functor is given by* $\mathfrak{T}\colon \mathbb{L}\text{-}\boldsymbol{GRP} \longrightarrow \boldsymbol{GRP}$.

Proof. Consider a group $(G,\cdot)$ and a family of mappings $(f_j\colon G \longrightarrow (H_j,\mu_j))_{j\in J}$, where each $f_j\colon G \longrightarrow H_j$ is a group homomorphism, μ_j is a subgroup of H_j, for each $j \in J$. Then the structure on μ on G is given by $\nu(g) = \bigwedge_j \mu_j(f_j(g))(= \bigwedge_j f_j^{\leftarrow}(\mu_j)(g))$, for all $g \in G$, note that for each $j \in J$, $f_j^{\leftarrow}(\mu_j)$ is also an $\mathbb{L}$-subgroup of G, and the arbitrary intersection ν is also an $\mathbb{L}$-subgroup of G, and hence $(G,\cdot,\nu) \in |\mathbb{L}\text{-}\mathbf{GRP}|$. Let $(Z,\cdot,\varrho) \in |\mathbb{L}\text{-}\mathbf{GRP}|$, we prove that the mapping $\varphi\colon (Z,\cdot,\varrho) \longrightarrow (G,\cdot,\nu)$ a group homomorphism is an $\mathbb{L}$-$\mathbf{GRP}$-morphism if and only if $f_j \circ \varphi\colon (Z,\cdot,\varrho) \longrightarrow (H_j,\cdot,\mu_j)$ is an $\mathbb{L}$-$\mathbf{GRP}$-morphism. We only show $g\colon (Z,\cdot,\varrho) \longrightarrow (G,\cdot,\nu)$ is an $\mathbb{L}$-$\mathbf{GRP}$-morphism. So, for any $z \in Z$, $\varrho(z) \leq \mu_j(f_j(\varphi(z)) = \bigwedge_{j\in J} f_j^{\leftarrow}(\mu_j)(\varphi(z)) = \nu(\varphi(z))$, i.e., $\varrho(z) \leq \nu(\varphi(z))$.

Theorem 2. *Let* $\mathbb{L} = (\mathbb{L}, \leq, * = \wedge)$ *be a complete Heyting algebra, and* $(G,\cdot,\nu)$ *be an* $\mathbb{L}$*-valued subgroup space and* $\mathcal{T}(G) = \{f\colon (G,\nu) \longrightarrow (G,\nu);\ f$ *is bijective and both* f *and* f^{-1} *satisfy* (‡) $\}$. *Then* $(\mathcal{T}(G),\cdot,\nabla)$ *is an* $\mathbb{L}$*-subgroup space, where* $(fg)(x) = f(x)g(x)$ *and* $f^{-1}(x) = (f(x))^{-1}$.

Proof. Clearly $(\mathcal{T}(G),\cdot)$ is a group under composition. Define
$\nabla(f) = \bigwedge_{x\in G}[\nu(x) \to \nu(f(x))], \forall f \in \mathcal{T}(G)$ (a)
and
$\nabla^{(-1)}(f) = \bigwedge_{x\in G}[\nu(x) \to \nu(f^{-1}(x))], \forall f \in \mathcal{T}(G)$ (b)
Combining (a) and (b) it follows upon using Proposition 1(7) that
$\nabla(f) = \bigwedge_{x\in G}[\nu(x) \to \nu(f(x)) \wedge \nu(f^{-1}(x))]$. Then clearly (LG1) and (LG3) are true upon using Proposition 1(7) and (LG2), i.e., $\nabla(id_G) = \top$, and $\nabla(f) \leq \nabla(f^{-1})$; we only look at (LG2).
For, let $f,g \in \mathcal{T}(X)$, then we have
$\nabla(f) \wedge \nabla(g) = \bigwedge_{x\in G}[\nu(x) \to \nu(f(x)) \wedge \nu(f^{-1}(x))] \wedge \bigwedge_{x\in G}[\nu(x) \to \nu(g(x)) \wedge \nu(g^{-1}(x))]$
$\leq \bigwedge_{x\in G}[\nu(x) \to \nu(f(x)) \wedge \nu(g(x)) \wedge \nu(g^{-1}(x)) \wedge \nu(f^{-1}(x))] \leq \bigwedge_{x\in G}[\nu(x) \to \nu(f(x)g(x)) \wedge \nu(g^{-1}(x)f^{-1}(x))]$
$= \bigwedge_{x\in G}[\nu(x) \to \nu(fg(x)) \wedge \nu((fg)^{-1}(x))] = \nabla(fg)$.

Definition 14. *[23, 25] An $\mathbb{L}$-valued subgroup is called* $\mathbb{L}$-valued normal subgroup *if for all $x, y \in G$ if it satisfies one of the following equivalent conditions:*
(1) $\nu(xy) = \nu(yx)$;
(2) $\nu(xyx^{-1}) \geq \nu(y)$;
(3) $\nu(xyx^{-1}) = \nu(y)$.

Definition 15. *A mapping $\ell \colon \mathbb{L}^X \longrightarrow \mathbb{L}^X$ is said to be an $\mathbb{L}$-valued*closure operation *on X if the following conditions hold for every $\nu, \mu \in \mathbb{L}^X$:*
(1) $\nu \leq \ell(\nu)$;
(2) $\nu \leq \mu$ implies $\ell(\nu) \leq \ell(\mu)$;
(3) $\ell(\ell(\nu)) = \ell(\nu)$;
(4) $\ell(\top_\emptyset) = \bot$.
The pair (X, ℓ) is called is called an $\mathbb{L}$-valued closure space *and $\nu \in \mathbb{L}^X$ is called* closed *if $\nu = \ell(\nu)$. Note that (2) implies $\ell(\nu) \vee \ell(\mu) \leq \ell(\nu \vee \mu)$, for any $\nu, \mu \in \mathbb{L}^X$.*
The category of all $\mathbb{L}$-valued closure spaces and all closure preserving mappings*, i.e., mappings $f \colon (X, \ell) \longrightarrow (Y, \ell)$ that satisfy $f^{\rightarrow}(\ell(\nu)) \leq \ell\left(f^{\rightarrow}(\nu)\right)$ for all $\nu \in \mathbb{L}^X$, is denoted by* $\mathbb{L}$-***CLS***.

Lemma 6. *We have the following forgetful functor forgetting $\mathbb{L}$-valued closure structure:*

$$\mathfrak{U} : \begin{cases} \mathbb{L}\text{-}\boldsymbol{CLS} & \longrightarrow & \boldsymbol{SET}(\mathbb{L}) \\ (X, \ell) & \longmapsto & (X, \nu) \\ f & \longmapsto & f \end{cases}$$

where $\mathfrak{U}((X, \ell)) = (X, \nu)$ and for $f \colon X \longrightarrow Y$, $\mathfrak{U}(f) = f$, $f^{\rightarrow} \colon \mathbb{L}^X \longrightarrow \mathbb{L}^Y$, and $\mathfrak{U}(f)$ yields an $\boldsymbol{SET}(\mathbb{L})$-morphism.

Let $X \in |\mathbf{SET}|$ and let $\Omega \subset \mathbb{L}^X$ be a collection of $\mathbb{L}$-subsets of X. Then we call Ω *a lattice-valued Moore collection* if every intersection of members of Ω belongs to Ω, i.e., given a family $(\nu_j)_{j \in J}$ of $\mathbb{L}$-subsets: $\forall j \in J,\ \nu_j \in \Omega \implies \bigwedge_{j \in J} \nu_j \in \Omega$. If Ω is a lattice-valued Moore collection containing $\top_\emptyset$, then if $\ell(\mu)_\Omega = \bigwedge\{\nu \in \Omega \colon \mu \leq \nu,\ \nu$ is $\mathbb{L}$-valued closed set$\}$, i.e. if $\ell(\mu)$ is the intersection of all $\mathbb{L}$-valued closed sets that contain μ, then ℓ is an $\mathbb{L}$-valued closure operator. We refer to Birkhoff [9], and Schechter [31], for the classical notion of Moore collection.

Example 4. *$\mathbb{L}$-valued subgroups of a group $(G, \cdot)$ form a lattice-valued Moore collection; this is so, since arbitrary intersection of $\mathbb{L}$-valued subgroups is again an $\mathbb{L}$-valued subgroup, cf. [11], pp. 115. In fact, if we let $\mu = \bigwedge_{j \in J} \nu_j$, then we can easily verify the Definition 13. In fact, (LG1) $\mu(e) = \bigwedge \nu_j(e) = \top$ for all $j \in J$; (LG2) upon using Proposition 1(10), we have: $\mu(x) * \mu(y) = (\bigwedge_{j \in J} \nu_j(x)) * (\bigwedge_{j \in J} \nu_j(y)) \leq \bigwedge_{j \in J} (\nu_j(x) * \nu_j(y)) \leq \bigwedge_{j \in J} \nu_j(xy) = \mu(xy)$, so, $\mu(x) * \mu(y) \leq \mu(xy)$; (LG3) $\mu = \bigwedge_{j \in J}(\nu_j(x)) \leq \bigwedge_{j \in J}(\nu_j(x^{-1})) = \mu(x^{-1})$. Also, if $\mu \in \mathbb{L}^G$, then $\mathbb{L}$-valued closure of μ is the subgroup generated by μ. This can be given as:*

$$\langle \ell(\mu) \rangle = \bigwedge\{\nu \colon \mu \leq \nu,\ \nu \text{ is closed } \mathbb{L}^G\text{-valued subgroup of } G\},$$

the $\mathbb{L}$-valued subgroup that contains μ.
In view of the Theorem 5.2.6[11], normal $\mathbb{L}$-valued subgroup of the group G form a lattice-valued Moore collection, and in particular, $\ell(\mu)$, $\mu \in \mathbb{L}^G$ is the normal $\mathbb{L}$-valued subgroup generated by μ. More precisely, $\langle \ell(\mu) \rangle = \bigwedge\{\nu \colon \mu \leq \nu,\ \nu$ is closed normal $\mathbb{L}^G$-valued subgroup of $G\}$,

Theorem 3. *$\mathbb{L}$-**CLS** is a topological category.*

Proof. Note that the objects of $\mathbb{L}$-**CLS** are structured sets and the composition of closure preserving mappings is closure preserving.
Consider X is a set, $\left(Y_j, \ell^j\right)_{j\in J}$ a family of $\mathbb{L}$-valued closure spaces and a source $\mathcal{S} = \left(f_j \colon X \longrightarrow (Y_j, \ell^j)\right)_{j\in J}$ of family of functions, then

$$\Omega = \{\omega \in \mathbb{L}^X \colon \omega = \textstyle\bigwedge_{j\in J} f_j^{\leftarrow}(\omega_j), \forall \omega_j = \ell_j(\omega_j),\ j \in J\}$$

is a lattice-valued Moore family which contains $\top_\emptyset$. Then Ω induces an $\mathbb{L}$-valued closure operation on X given by: $\ell(\mu)_\Omega = \bigwedge\{\omega \in \Omega \colon \mu \leq \omega\}$, for all $\mu \in \mathbb{L}^X$. Now let $(Z, \ell) \in |\mathbb{L}\text{-}\mathbf{CLS}|$, and $g \colon Z \longrightarrow X$ be a function such that $f_j \circ g \colon (Z, \ell) \longrightarrow (Y, \ell_j)$ is closure preserving mapping for all $j \in J$. If $\mu \in \mathbb{L}^X$ is a ${}^{-\Omega}$ closed, then $\mu \in \Omega$ and thus $\mu = \bigwedge_{j\in J} f_j^{\leftarrow}(\omega_j)$ where $\omega_j = \ell_j(\omega_j)$ in (Y_j, ℓ_j). In view of Proposition 1.2(5) [22], we have:

$$g^{\leftarrow}(\mu) = g^{\leftarrow}\left(\bigwedge_j f_j^{\leftarrow}(\omega_j)\right) = \bigwedge_j g^{\leftarrow}\left(f_j^{\leftarrow}(\omega_j)\right) = \bigwedge_j (f_j \circ g)^{\leftarrow}(\omega_j)$$

This implies $(f_j \circ g)^{\leftarrow}(\omega_j)$ is closed in (Z, ℓ) implying $g^{\leftarrow}(\mu)$ is closed in (Z, ℓ).

Remark 5. *Every $\mathbb{L}$-valued topological space (X, Δ) is an $\mathbb{L}$-valued closure space with the closure operation defined by: $\ell(\nu) = \overline{\nu}^{(X,\Delta)} = \overline{\nu}^X$ for every $\nu \in \mathbb{L}^X$. Also, every mapping $f \colon (X, \Delta) \longrightarrow (Y, \Gamma)$ continuous if and only if it is closure preserving with respect to the induced $\mathbb{L}$-valued closure operations. In fact, if $\nu \in \mathbb{L}^X$, then in view of the Proposition 1.4 [22], $f^{\rightarrow}(\ell(\nu)) = f^{\rightarrow}(\overline{\nu}^X) \leq \overline{f^{\rightarrow}(\nu)}^Y = \ell(f^{\rightarrow}(\nu))$, i.e., $f^{\rightarrow}(\ell(\nu)) \leq \ell(f^{\rightarrow}(\nu))$, meaning f is closure preserving. Conversely, let $\nu \in \mathbb{L}^X$ and f be closure preserving, then $f^{\rightarrow}(\overline{\nu}^X) = f^{\rightarrow}(\ell(\nu)) \leq \ell(f^{\rightarrow}(\nu)) = \overline{f^{\rightarrow}(\nu)}^Y$, i.e., $f^{\rightarrow}(\overline{\nu}^X) \leq \overline{f^{\rightarrow}(\nu)}^Y$ meaning the mapping $f \colon (X, \Delta) \longrightarrow (Y, \Gamma)$ is continuous by the Proposition 1.4 [22]. Thus we have the following.*

Corollary 1. *$\mathbb{L}$-**TOP**, the category of $\mathbb{L}$-valued topological spaces and continuous mappings is a full subcategory of the category $\mathbb{L}$-**CLS***

Definition 16. *A triple $(G, \cdot, \ell)$ is called an* $\mathbb{L}$-closure group *if $(G, \cdot) \in |\boldsymbol{GRP}|$ and $(G, \ell) \in |\mathbb{L}\text{-}\boldsymbol{CLS}|$ such the following are fulfilled:*
(clGM) $\ell(\nu)(x) * \ell(\nu)(y) \leq \ell(\nu \cdot \nu)(xy)$, $\forall \nu \in \mathbb{L}^G$ and $\forall x, y \in G$;
(clGI) $\ell(\nu)(x) \leq \ell(\nu^{-1})(x^{-1})$, $\forall \nu \in \mathbb{L}^G$ and $x \in G$.
*The category of all $\mathbb{L}$-valued closure groups and closure-preserving group homomorphisms is denoted by $\mathbb{L}$-**CLGRP**.*

Remark 6. *If we consider each $\nu \in \mathbb{L}(G)$, i.e., each $\nu \in \mathbb{L}^G$ is an $\mathbb{L}$-valued subgroup of the group G, then we obtain a category $\mathbb{L}$-**CLGRP*** of all $\mathbb{L}$-valued closure of $\mathbb{L}$-valued subgroups of G, and closure-preserving mappings. Then $\mathbb{L}$-**CLGRP*** is a subcategory of $\mathbb{L}$-**CLGRP**.*

Theorem 4. *$\mathbb{L}$-**CLGRP** is a topological category.*

Proof. Consider $(G, \cdot)$ a group, and a source $\mathcal{S} = (f_j : (G, \cdot) \longrightarrow (G_j, \cdot, \ell_j))_{j\in J}$ of family of functions, where for each $j \in J$, $f_j : G \longrightarrow G_j$ is a group homomorphism, then

$$\Omega = \{\omega \in \mathbb{L}^G : \omega = \bigwedge_{j\in J} f_j^{\leftarrow}(\omega_j), \forall \omega_j = \ell_j(\omega_j),\ j \in J\}$$

In view of Theorem 3, we have $(G, \cdot, \ell)$ is an $\mathbb{L}$-valued closure space. We only verify (clGM). So we have:
$\ell(\omega)(x) * \ell(\omega)(y) = \bigwedge_{j\in J} f_j^{\leftarrow}(\omega_j)(x) * \bigwedge_{j\in J} f_j^{\leftarrow}(\omega_j)(y) \leq \bigwedge_{j\in J} f_j^{\leftarrow}(\omega_j) \odot f_j^{\leftarrow}(\omega_j)(xy)$
$= \bigwedge_{j\in J} f_j^{\leftarrow}(\omega_j \odot \omega_j)(xy) \leq \ell(\omega \cdot \omega)(xy)$.

Definition 17. *[8, 19, 32] An $\mathbb{L}$-tolerance space is a pair (X, τ), where $\tau : X \times X \longrightarrow \mathbb{L}$ such that*
(T1) $\tau(x, x) = \top$, $\forall x \in X$ (reflexivity);
(T2) $\tau(x, y) = \tau(y, x)$ (symmetry).
*If, in addition τ satisfies (T3) $\tau(x, y) * \tau(y, z) \leq \tau(x, z)$, for any $x, y, z \in X$, then we speak of transitive tolerance relation which is essentially gives an $\mathbb{L}$-equivalence relation. A mapping between $\mathbb{L}$-valued tolerance spaces (resp. transitive $\mathbb{L}$-valued tolerance spaces): $f : (X, \tau) \longrightarrow (Y, \tau')$ is called $\mathbb{L}$-valued tolerance preserving if $\tau(x, y) \leq \tau'(f(x), f(y))$. The category of all $\mathbb{L}$-valued tolerance spaces and $\mathbb{L}$-tolerance preserving mappings is denoted by $\mathbb{L}$-**TOL** while $\mathbb{L}$-**TranTOL** denotes the category of transitive $\mathbb{L}$-tolerance spaces.*

For an **MV**-valued algebra $\mathbb{L}$, given $\mathbb{L}$-**TranTOL** a category of transitive $\mathbb{L}$-valued tolerance spaces and $\mathbb{L}$-valued tolerance preserving mappings, one can obtain a functor $\mathcal{A} : \mathbb{L}$-**TOL**$\longrightarrow \mathbb{L}$-**SET** where $\mathcal{A}(X, \tau) = (X, \tau\mathbb{D})$, $\mathbb{D} : X \longrightarrow X \times X$ and $\mathcal{A}(f) = f$, here $\mathcal{A}(f)$ sends f to an $\mathbb{L}$-tolerance preserving mapping to $f : (X, \tau\mathbb{D}) \longrightarrow (Y, \tau'\mathbb{D})$, i.e., $\tau\mathbb{D}(x) = \tau(x, x) \leq \tau'(f(x), f(x)) = \tau'\mathbb{D}(f(x))$, i.e., $\tau\mathbb{D}(x) \leq \tau'\mathbb{D}(f(x))$. Conversely, given $\mathbb{L}$-**SET**, one obtains a functor $\mathcal{B} : \mathbb{L}$-**SET**$\longrightarrow \mathbb{L}$-**TranTOL** as defined by: $\mathcal{B}(X, \nu) = (X, \tau := \nu \wedge \nu)$ and $\mathcal{B}(f) = f$, $\tau(x, y) = \nu(x) \wedge \nu(y) \leq \nu(f(x)) \wedge \nu(f(y)) = \tau(f(x), f(y))$.
In view of [11], pp 148, for a group $(G, \cdot)$, we consider a mapping $\varrho_L : \mathbb{L}^G \longrightarrow \mathbb{L}^{G\times G}$ defined by: $\varrho_L(\nu)(x, y) = \nu(x^{-1}y)$, and analogously, $\varrho_R(\nu)(x, y) = \nu(xy^{-1})$. Then we have the following.

Lemma 7. *Let $(G, \cdot) \in |$**GRP**$|$, and the category $\mathbb{L}$-**TranTOL** consists of morphisms $f : (G, \tau) \longrightarrow (H, \varrho')$ which are $\mathbb{L}$-valued tolerance preserving such that each morphism is a group homomorphism. Then*

$$\mathfrak{A} : \begin{cases} \mathbb{L}\text{-}\boldsymbol{GRP} & \longrightarrow & \mathbb{L} - \boldsymbol{TranTOL} \\ (G, \nu) & \longmapsto & (G, \varrho_L(\nu)) \\ f & \longmapsto & f \end{cases}$$

Proof. Let $\nu \in \mathbb{L}(G)$, then we have $\rho_L(\nu)(x,x) = \nu(x^{-1}x) = \nu(e) = \top$ which is (T1); for (T2), we apply Theorem 5.1.1(5)[11](see also, Theorem 1.2.2[24]) to get $\rho_L(\nu)(x,y) = \nu(x^{-1}y) = \nu((x^{-1}y)^{-1}) = \nu(y^{-1}x) = \rho_L(y,x)$. Now for any $x,y,z \in X$, $\rho_L(\nu)(x,y) * \rho_L(y,z) = \nu(x^{-1}y) * \nu(y^{-1}z) \leq \nu(x^{-1}yy^{-1}z) = \nu(x^{-1}z) = \rho_L(\nu)(x,z)$, which is (T3). To check the morphism part, we have for any $x,y \in G$ and $\nu \in \mathbb{L}(G)$: $\tau(x,y) = \rho_L(\nu)(x,y) = \nu(x^{-1}y) \leq \nu'(f(x^{-1}y)) = \nu'((f(x))^{-1}f(y)) = \rho_L(\nu)(f(x),f(y)) = \tau'(f(x),f(y))$, i.e., $\tau(x,y) \leq \tau'(\nu')(f(x),f(y))$.

Lemma 8. *Let $(G,\cdot) \in |\boldsymbol{GRP}|$, and the category $\mathbb{L}$-$\boldsymbol{TranTOL}$ consists of morphisms $f\colon (G,\rho_L(\nu)) \longrightarrow (H,\rho_L(\nu'))$ which are $\mathbb{L}$-valued tolerance preserving such that each morphism is a group homomorphism. Then*

$$\mathfrak{B} : \begin{cases} \mathbb{L} - \boldsymbol{TranTOL} & \longrightarrow & \mathbb{L}\text{-}\boldsymbol{GRP} \\ (G, \varrho_L(\nu)) & \longmapsto & (G,\nu) \\ f & \longmapsto & f \end{cases}$$

Proof. Let $\nu \in \mathbb{L}^G$, and $(G,\rho_L(\nu)) \in |\mathbb{L}\text{-}\mathbf{TranTOL}|$, it suffices to show that $\nu \in \mathbb{L}(G)$. Thus, for any $x \in X$, $\nu(e) = \nu(x^{-1}x) = \rho_L(\nu)(x,x) = \top$ which is (LG1). For (LG2) is obviously true while for (LG3), we have for any $x,y \in G$: $\nu(x) * \nu(y) = \nu(xe) * \nu(ey) = \rho_L(\nu)(x,e) * \rho_L(\nu)(e,y) \leq \rho_L(\nu)(x,y) = \nu(x^{-1}y)$, i.e., $\nu(x) * \nu(y) \leq \nu(x^{-1}y)$, this happens when we combine (LG2) and (LG3), cf. Theorem 5.1.3[11]. This shows that $\nu \in \mathbb{L}(G)$. For the morphism part, let $x \in G$ and $\nu \in \mathbb{L}^G$. Then $\nu(x) = \nu(ex) = \rho_L(\nu)(e,x) \leq \rho_L(\nu')(f(e),f(x)) = \nu'\left((f(e))^{-1}f(x)\right) = \nu'(f(ex)) = \nu'(f(x))$, i.e., $\nu(x) \leq \nu'(f(x))$.

5. Enriched latticed-valued subgroups on lattice-valued neighborhood groups

Let $\mathbb{L} = (\mathbb{L}, \leq, *)$ be a complete $\mathbf{MV}$-valued algebra with square roots. If $\left(X, \mathfrak{N} = (\mathfrak{N}_x)_{x\in X}\right)$ is a stratified $\mathbb{L}$-neighborhood space, then in view of [12] (page 13), and [18] (page 226), one can see that $\mathfrak{N}$ induces a closure operator $^{-}\colon \mathbb{L}^X \longrightarrow \mathbb{L}^X$ given for any $x \in X$ and $\nu \in \mathbb{L}^X$ by

$$\overline{\nu}(x) = ([\mathfrak{N}_x](\nu \to \bot)) \to \bot.$$

Theorem 5. *[3, 12, 18] (a) Let $\left(X, \mathfrak{N} = (\mathfrak{N}_x)_{x\in X}\right) \in |\boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{NS}|$. Then*

$$\overline{\nu}(x) = \bigvee\{\mathcal{F}(\nu)\colon\ \mathcal{F} \in \mathcal{F}^s_{\mathbb{L}}(X),\ \mathcal{F} \geq \mathfrak{N}_x\},\ \forall \nu \in \mathbb{L}^X,\ \textit{and}\ \forall x \in X.$$

(b) Let $\left(G, \cdot, \mathfrak{N} = (\mathfrak{N}_x)_{x\in G}\right) \in |\boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{NGRP}|$ and $\nu \in \mathbb{L}^G$ be an $\mathbb{L}$-valued subgroup of a group G. Then the $\mathbb{L}$-valued closure $\overline{\nu}$ of ν in (a) is an $\mathbb{L}$-valued subgroup of G.
(c) Let $(G,\cdot,\mathfrak{N}) \longrightarrow (H,\cdot,\mathfrak{M})$ be continuous group homomorphism. Then $\overline{\nu}(x) \leq \overline{f^{\rightarrow}(\nu)}(f(x))$ for all $\nu \in \mathbb{L}^G$ and $x \in G$. Moreover, if $\nu \in \mathbb{L}^G$ is an $\mathbb{L}$-valued subgroup of G, then $\overline{f^{\rightarrow}(\nu)}$ is an $\mathbb{L}$-valued subgroup of H, and if $\mu \in \mathbb{L}^G$ is a $\mathbb{L}$-valued subgroup of H, then $\overline{f^{\leftarrow}(\mu)}$ is an $\mathbb{L}$-valued subgroup of G.
(d) If $\nu \in \mathbb{L}^G$ is an $\mathbb{L}$-valued normal subgroup of a group G, then $\overline{\nu}$ is also an $\mathbb{L}$-valued

normal subgroup of G.
(e) If $(G,\cdot,\mathfrak{N}) \longrightarrow (H,\cdot,\mathfrak{M})$ is a continuous group homomorphism and $\mu \in \mathbb{L}^H$ is an $\mathbb{L}$-valued subgroup of H, then $\overline{f^{\leftarrow}(\mu)}$ is an $\mathbb{L}$-valued subgroup of G.

Proof. (b) follows from the Theorem 5.1[3].
(c) Let $\nu \in \mathbb{L}^G$, and $x \in G$. Then since $\nu \leq f^{\leftarrow}(f^{\rightarrow}(\nu))$ due to Definition 6 (LF2), $\mathcal{F}(\nu) \leq \mathcal{F}\left(f^{\leftarrow}(f^{\rightarrow}(\nu))\right) = f^{\Rightarrow}(\mathcal{F})(f^{\rightarrow}(\nu))$, and since $\mathfrak{M}_{f(x)} \leq f^{\Rightarrow}(\mathfrak{N}_x)$ due to continuity of f, we have
$\overline{\nu}(x) = \bigvee\{\mathcal{F}(\nu)\colon \mathcal{F} \in \mathcal{F}^s_{\mathbb{L}}(X), \mathcal{F} \geq \mathfrak{N}_x\} \leq \bigvee\{f^{\Rightarrow}(\mathcal{F})(f^{\rightarrow}(\nu))\colon f^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}^s_{\mathbb{L}}(Y), f^{\Rightarrow}(\mathcal{F})(f^{\rightarrow}(\nu)) \geq f^{\Rightarrow}(\mathfrak{N}_x)(f^{\rightarrow}(\nu))\}$
$\leq \bigvee\{f^{\Rightarrow}(\mathcal{F})(f^{\rightarrow}(\nu))\colon f^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}^s_{\mathbb{L}}(H), f^{\Rightarrow}(\mathcal{F})(f^{\rightarrow}(\nu)) \geq \mathfrak{M}_{f(x)}(f^{\rightarrow}(\nu))\}$
$= \bigvee\{\mathcal{G}(f^{\rightarrow}(\nu))\colon \mathcal{G} \in \mathcal{F}^s_{\mathbb{L}}(Y),\ \mathcal{G} \geq \mathfrak{M}_{f(x)}\} = \overline{f^{\rightarrow}(\nu)}(f(x))$, i.e., $\overline{\nu}(x) \leq \overline{f^{\rightarrow}(\nu)}(f(x))$.
(d) Let $\nu \in \mathbb{L}^G$ be an $\mathbb{L}$-valued normal subgroup of a group G, and consider the mapping $\mathcal{C}_a\colon G \longrightarrow G$ defined by $\mathcal{C}_a(g) = a^{-1}ga$; need to that $\overline{\nu}$ is also an $\mathbb{L}$-normal subgroup of G. Note that ν is $\mathbb{L}$-normal subgroup of G if and only if $\nu(aga^{-1}) = \nu(g)$. Now since the mapping $\mathcal{C}_a$ is continuous, we have $\overline{\nu}(g) \leq \overline{\mathcal{C}_a(\nu)}(\mathcal{C}_a(g)) = \bigvee_{x\in\mathcal{C}_a^{\leftarrow}(g)} \overline{\nu}(x) = \bigvee_{\mathcal{C}_a(x)=g} \overline{\nu}(x) = \overline{\nu}(aga^{-1})$, i.e., $\overline{\nu}(aga^{-1}) \geq \overline{\nu}(g)$, meaning that $\overline{\nu}$ is a normal $\mathbb{L}$-valued subgroup of G.
(e) Let $(G,\cdot,\mathfrak{N}) \longrightarrow (H,\cdot,\mathfrak{M})$ be a continuous group homomorphism, and $\mu \in \mathbb{L}(H)$. Then
$\overline{f^{\leftarrow}(\mu)}(e) = \bigvee\{\mathcal{F}(f^{\leftarrow}(\mu))\colon \mathcal{F} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathcal{F} \geq \mathfrak{N}^e\}$
$\geq \bigvee\{[e](f^{\leftarrow}(\mu)\colon [e] \in \mathcal{F}^s_{\mathbb{L}}(G), [e] \geq \mathfrak{N}^e\}$
$\geq \bigvee\{\mu(e)\colon [e] \in \mathcal{F}^s_{\mathbb{L}}(G), [e] \geq \mathfrak{N}^e\} = \top$, whence $\mu(e) = \top$, since $\mu \in \mathbb{L}(G)$, implying $\overline{f^{\leftarrow}(\mu)}(e) = \top$.
Now let $x, y \in G$ and $\mu \in \mathbb{L}^H$. Then in view of the Definition 1(GLM3), we have:
$\overline{f^{\leftarrow}(\mu)}(x) * \overline{f^{\leftarrow}(\mu)}(y) = \bigvee\{\mathcal{F}(f^{\leftarrow}(\mu))\colon \mathcal{F} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathcal{F} \geq \mathfrak{N}^x\} * \bigvee\{\mathcal{G}(f^{\leftarrow}(\mu))\colon \mathcal{G} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathcal{G} \geq \mathfrak{N}^y\}$
$= \bigvee\{\mathcal{F}(f^{\leftarrow}(\mu)) * \mathcal{G}(f^{\leftarrow}(\mu))\colon \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathcal{F} \geq \mathfrak{N}^x, \mathcal{G} \geq \mathfrak{N}^y\}$
$\leq \bigvee\{\mathcal{F} \odot \mathcal{G}(f^{\leftarrow}(\mu))\colon \mathcal{F} \odot \mathcal{G} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathcal{F} \odot \mathcal{G} \geq \mathfrak{N}_x \odot \mathfrak{N}^y\}$ (By applying Theorem 1.2.8 and Theorem 1.2.11[24], whence $f^{\leftarrow}(\mu) \in \mathbb{L}(G)$)
$\leq \bigvee\{\mathcal{F} \odot \mathcal{G}(f^{\leftarrow}(\mu))\colon \mathcal{F} \odot \mathcal{G} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathfrak{N}_{xy} \leq \mathcal{F} \odot \mathcal{G}\}$ (since $(G,\cdot,\mathfrak{N}) \in |\mathbf{S}\mathbb{L}\text{-}\mathbf{NS}|$, applying the Definition 10(LNGM), and due to Lemma 1, $\mathcal{F} \odot \mathcal{G} \in \mathcal{F}^s_{\mathbb{L}}(G)$)
$= \bigvee\{\mathcal{H}(f^{\leftarrow}(\mu))\colon \mathcal{H} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathfrak{N}_{xy} \leq \mathcal{H}\} = \overline{f^{\leftarrow}(\mu)}(xy)$.
Finally, since $\overline{f^{\leftarrow}(\mu)} \geq f^{\leftarrow}(\mu)$, we get $\overline{f^{\leftarrow}(\mu)}(x^{-1}) \geq \overline{f^{\leftarrow}(\mu)}(x)$, for any $x \in G$. In fact, for any $\mu \in \mathbb{L}(H)$, $f^{\leftarrow}(\mu) \in \mathbb{L}(G)$ by Theorem 1.2.11[24]. So, we have:
$\overline{f^{\leftarrow}(\mu)}(x) = \bigvee\{\mathcal{F}(f^{\leftarrow}(\mu))\colon \mathcal{F} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathcal{F} \geq \mathfrak{N}^x\}$
$\leq \bigvee\{\mathcal{F}^{-1}(f^{\leftarrow}(\mu))\colon \mathcal{F}^{-1} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathcal{F}^{-1} \geq (\mathfrak{N}^x)^{-1}\}$
$\leq \bigvee\{\mathcal{F}^{-1}(f^{\leftarrow}(\mu))\colon \mathcal{F}^{-1} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathcal{F}^{-1} \geq \mathfrak{N}^{x^{-1}}\}$ (by Definition 10(LNGI))
$= \bigvee\{\mathcal{G}(f^{\leftarrow}(\mu))\colon \mathcal{G} \in \mathcal{F}^s_{\mathbb{L}}(G), \mathcal{G} \geq \mathfrak{N}^{x^{-1}}\}$
$= \overline{f^{\leftarrow}(\mu)}(x^{-1})$.

Lemma 9. *[2] Let $(G,\cdot,\Delta) \in |\boldsymbol{S}\mathbb{L}\text{-}\boldsymbol{TOPGRP}|$, $\mu \in \Delta$ and $\nu \in \mathbb{L}^G$. Then $\mu \cdot \nu \in \Delta$.*

Proof. Let $x \in G$, $\mu \in \Delta$ and $\nu \in \mathbb{L}^G$. Then $\mu \cdot \nu(x) = \bigvee_{st=x} \mu(x) * \nu(t) = \bigvee_{t\in G} \mu(xt^{-1}) * \nu(t) = \bigvee_{t\in G} \mathcal{R}_t(\mu)(x) * \nu(t)$. Fix $t \in G$, then $\nu(t)$ is constant and

$\nu(t) \in \mathbb{L}$. Since and $\mathcal{R}_t \colon G \longrightarrow G$ is a homeomorphism, and $\mu \in \Delta$, $\bigvee_{t\in G} \mathcal{R}_t(\mu) \in \Delta$ and since Δ is stratified, and $(\mathbb{L}, *)$ is commutative semigroup, we have $\bigvee_{t\in G} \mathcal{R}_t(\mu) * \nu(t) = \nu(t) * \bigvee_{t\in G} \mathcal{L}_t(\mu) \in \Delta$, i.e., $\mu \cdot \nu \in \Delta$.

Proposition 3. *[18] Let $(X, \Delta_{\mathfrak{N}})$ be a stratified $\mathbb{L}$-valued topological space with a corresponding stratified $\mathbb{L}$-valued neighborhood system $\mathfrak{N}$. Then $(X, \Delta_{\mathfrak{N}})$ is Hausdorff-separated if and only if for all $x \neq y \in X$ there are $\nu_1, \nu_2 \in \Delta_{\mathfrak{N}}$ such that $\nu_1 * \nu_2 = \top_{\emptyset}$ and $\nu_1(x) * \nu_2(y) \neq \bot$.*

Definition 18. *[18] Let (X, Δ) be a stratified $\mathbb{L}$-valued topological space, $\mathfrak{N} = (\mathfrak{N}_x)_{x\in X}$ be the corresponding $\mathbb{L}$-valued neighborhood system, and A be a subset of X. Then closure of A, written as $\overline{A}$, is given by*

$$\overline{A} = \{x \in X \colon \mathfrak{N}_x(\top_{X\cap A^c}) = \bot\}$$

A subset of X is said to be closed *with respect to Δ if $A = \overline{A}$.*

Lemma 10. *A stratified $\mathbb{L}$-valued topological group $(G, \cdot, \Delta_{\mathfrak{N}})$ is Hausdorff-separated if and only if some singleton $\{a\} \subseteq G$ is closed. In particular $\{e\}$ is a closed subgroup of G.*

Proof. Let $\{a\} \subseteq G$ be closed subset of G. Then since the mapping $\varphi \colon (G\times G, \Delta\times\Delta) \to (G, \Delta), (g, h) \longmapsto g^{-1}ha$ is continuous, we have $\varphi^{-1}(\{a\}) = \{(g, g) \colon g \in G\} \subseteq G \times G$, the diagonal which in view of the Corollary 6.2.1.2 [18], is a closed subset of $G\times G$ with respect to the product stratified $\mathbb{L}$-topology $\Delta \times \Delta$ implying that $(G, \cdot, \Delta)$ is Hausdorff-separated. Conversely, let $x \notin \{a\}$. Then $x \neq a \in X$ yields that there are $\nu_1, \nu_2 \in \Delta$ such that $\nu_1 * \nu_2 \leq \top_{X\cap\{a\}^c}$ and $\mathfrak{N}_x(\nu_1) * \mathfrak{N}_a(\nu_2) \neq \bot$, which implies that $x \notin \overline{\{a\}}$.

Lemma 11. *If $(G, \cdot, \Delta_{\mathfrak{N}})$ is a Hausdorff-separated stratified $\mathbb{L}$-valued topological group, and A be a closed subgroup of G, then the normalizer of A in G: $\mathsf{N}_G(A) = \{g \in G \colon \gamma_a(A) = A\}$ is a closed subgroup of G, where $\gamma_a \colon G \longrightarrow G$ defined by $\gamma_a(g) = ag^{-1}a$ the conjugation map.*

Proof. If $a \in A$, take $c_a(g) = gag^{-1}$. Then the mapping $c_a \colon G \longrightarrow G$ is continuous and hence the inverse image of the closed set A: $c_a^{-1}(A) = \{g \in G \colon gag^{-1} \in A\}$ is closed. Thus, we have

$$B := \bigwedge_{a\in A} c_a^{-1}(A) = \{g \in G \colon \gamma_a(A) \subseteq A\}$$

is a closed subset of G. Since the inversion mapping $\jmath \colon G \longrightarrow G, g \longmapsto g^{-1}$ is a homeomorphism, A^{-1} is closed, since A is closed, and hence $\mathsf{N}_G(A) = B \cap A^{-1}$ is closed.

Lemma 12. *Let $(G, \cdot, \Delta)$ be a stratified $\mathbb{L}$-valued topological group, $\mathfrak{N}$ be a corresponding stratified $\mathbb{L}$-valued neighborhood system on G and A is a subset of G. Then the centralizer*

$$\mathsf{Z}_G(A) = \{g \in G \colon [g, a] = e \quad \forall a \in A\}$$

is closed with respect to Δ. In particular, the center of G is closed subgroup.

Proof. If $a \in A$, then the mapping $\varphi : G \longrightarrow G, g \longmapsto [g, a] = gag^{-1}a^{-1}$ is continuous, where the element of the type $gag^{-1}a^{-1}$ is called *commutator* of the group G. Now since $\{e\}$ is closed subset of G, and since the inverse image of closed subsets under continuous mapping are again closed, in view of the Corollary 6.2.1.2 [18], $\mathsf{Z}_G(a) = \{g \in G : [g, a] = e\}$ is closed, and as the $\mathsf{Z}_G(A) = \bigwedge_{a \in A} \mathsf{Z}_G(a)$ is closed, hence the result follows.

6. Conclusion

In this article, as a continuation of our previous work on $\mathbb{L}$-valued topological groups, where the underlying lattice $\mathbb{L}$ was an enriched *cl*-premonoid, we have presented two types of results, one is about the relationship between $\mathbb{L}$-valued topological groups and their corresponding Kent convergence groups and conversely; the other is about $\mathbb{L}$-valued closure of $\mathbb{L}$-valued subgroup of a group. Although, it is an well-known fact that there is a close connection between principal limit convergence spaces and closure spaces, but we did not touch upon this issue here even for $\mathbb{L}$-valued generalization of these structures in conjunction with group structures, that is, to study $\mathbb{L}$-valued principal convergence spaces and $\mathbb{L}$-valued closure spaces. We intend to look into this issue in a future paper.

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