



On the Continuous Action of Enriched Lattice-Valued Convergence Groups: Some Examples

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Abstract. Starting with a category $SL\text{-CONVGRP}$, of stratified enriched cl -premonoid-valued convergence groups as introduced earlier, we present a category $SL\text{-CONVTGRP}$, of stratified enriched cl -premonoid-valued convergence transformation groups, the idea behind this category is crept in the notion of convergence transformation group - a generalization of topological transformation group. In this respect, we are able to provide natural examples in support to our endeavor; these examples, however, stem from the action of convergence approach groups on convergence approach spaces, and the action of probabilistic convergence groups under triangular norm on probabilistic convergence spaces. Based on the category of enriched lattice-valued convergence spaces, a Cartesian closed category that enjoys lattice-valued convergence structure on function space, we look into among others, the lattice-valued convergence structures on the group of homeomorphisms of enriched lattice-valued convergence spaces, generalizing a concept of convergence transformation groups on convergence spaces, obtaining a characterization.

1. Introduction

Considering the notion of lattice-valued filter as introduced in [28], Jäger studied the category of lattice-valued convergence spaces, $L\text{-CONV}$, where it is pointed out, one may put it as: the notion of lattice-valued convergence space is an extension of $\{0, 1\}$ -valued convergence space which can be identified with classical convergence space (cf. Remark in pp. 6 [30], and [45]). He showed among other results that $L\text{-CONV}$ is a Cartesian closed category [30] (see also, [1, 32]), stimulating interest among many researchers to work on lattice-valued convergence spaces, and quite a good number of papers surfaced in recent years, we quote here a few of them, cf. [2, 3, 6, 16, 19, 20, 25, 31–33, 35, 41, 51]. It may be mentioned here that the importance of various types of convergence, particularly, filter-theoretic convergence structures contributed immensely for the development of set-theoretic topology in general, and functional analysis in particular [8, 14, 15, 29, 50].

In 1988, Lowen and Lowen introduced a category of convergence approach spaces, CAP [37], which is also

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This article is dedicated to Professor Eva Colebunders on her 70th Birthday

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a Cartesian closed category. In [31], it is proved that **CAP** is simultaneously a reflective and coreflective subcategory of the category **L-CONV**. Considering the category **CAP**, Colebunders et al in [17] brought to light a category **CAM**, of convergence approach monoids, and studied the continuous action of convergence approach monoids on convergence approach spaces; with the help of the notion of convergence approach groups [38], they also studied the continuous action of convergence approach groups on convergence approach spaces. We, on the other hand, in [3] (see also [2]) introduced and studied the notion of stratified enriched cl -premonoid-valued convergence groups with the help of the notion of lattice-valued convergence structure introduced in [30].

The theory of topological transformation groups is quite old, and forms a fascinating and wide-ranging topic in the realm of mathematics having enormous applications, such as, topological dynamics, abstract harmonic analysis, ergodic theory, geometry, Lie groups, differential equations to name a few (see f.i. [13, 23, 40]). Although the theory of convergence transformation groups is relatively new, but possesses appealing characteristic, and thus, we believe that there is ample opportunity to do research in this direction including our efforts to their extensions to lattice-valued cases. We refer to [48] for further explanations of the importance of studying classical convergence structures to group action.

In [21] (see also [13, 43]), Gevorgyan while studying the notion of so-called G -spaces in connection to the action of topological groups on topological spaces, pointed out that the group of self-homeomorphisms cannot usually be made into a topological group unless the topological spaces under consideration are locally compact and locally connected. A similar argument made by Park in [42] who originally studied a notion of an action of convergence group on a convergence space leading to a notion of convergence transformation group on a convergence space, argued that a meaningful results in topological transformation groups could be achieved when one considers topological spaces as locally compact and Hausdorff, cf. [13]. He further argued that if one considers the category of convergence spaces [9–12, 44, 46, 47], then situation turned out to be very simple, needs no mention of an extra property. Note that the investigation of classical notion of group of homeomorphisms among other notable authors goes back to Arens [7], see also [18, 39, 40].

The motive behind the present article is, first, to introduce a notion of action of stratified enriched cl -premonoid-valued convergence group on enriched cl -premonoid-valued convergence space, and provide two important classes of natural examples. Secondly, we investigate stratified lattice-valued convergence structure on group of homeomorphisms, and provide a characterization. Thirdly, given an arbitrary group and an enriched lattice-valued convergence structure on it, we give a procedure to construct an enriched lattice-valued convergence transformation group on the given enriched lattice-valued convergence space. Furthermore, we look into categorical connection between the concepts of stratified enriched lattice-valued convergence transformation groups and the category of convergence approach transformation groups, and also, category of probabilistic convergence transformation groups under triangular norms. We arrange our work as follows.

In Preliminary Section 2, we give a general view of enriched cl -premonoid lattice structures including the well-known notions of enriched lattice-valued convergence structure that will be needed in the sequel. In Section 3, we provide the notion of the continuous action of enriched lattice-valued convergence groups on enriched lattice-valued convergence spaces leading to the notion of enriched lattice-valued convergence transformation groups on enriched lattice-valued convergence spaces, generalizing the notion of convergence transformation groups on convergence spaces - a classical notion that was introduced for the first time by Park in [42]; here we study group of homeomorphisms of enriched lattice-valued convergence spaces. Moreover, we present here a characterization theorem on lattice-valued convergence transformation groups. Section 4 deals with a construction of a stratified convergence transformation group while in Section 5, we bring the idea of convergence approach transformation group on convergence approach space - an idea which has not been mentioned explicitly in [17]; in this section we explore a possible link between the categories **SL-CONVTGRP**, of stratified enriched lattice-valued convergence transformation groups, and **CAPTGRP**, of convergence approach transformation groups. In Section 6, introducing a notion of probabilistic convergence transformation group under triangular norm [34], we provide another class of examples of enriched lattice-valued convergence transformation groups, here we explore again the relation between the categories **SL-CONVTGRP** and **PCONVTGRP***, of probabilistic convergence transformation

groups under triangular norm.

2. Preliminaries

Throughout the text we consider $L = (L, \leq)$ a complete lattice with \top , the top element and \perp , the bottom element of L , for further details cf. [22, 49].

Definition 2.1. [24] A triple $(L, \leq, *)$, where $*$: $L \times L \rightarrow L$ is a binary operation on L , is called a **GL-monoid** if and only if the following holds:

(GLM1) $(L, *)$ is a commutative semigroup;

(GLM2) $\forall \alpha \in L: \alpha * \top = \alpha$;

(GLM3) $*$ is distributive over arbitrary joins:

$\gamma * (\bigvee_{k \in K} \alpha_k) = \bigvee_{k \in K} (\gamma * \alpha_k)$, for $k \in K, \alpha_k, \gamma \in L$;

(GLM4) for every $\gamma \leq \alpha$ there exists $\beta \in L$ such that $\gamma = \alpha * \beta$ (divisibility).

Note that (GM1), (GM2) and (GM3) mean that we have a commutative and *integral* quantale. If $*$ = \wedge , then the triple (L, \leq, \wedge) is called a frame or a complete Heyting algebra.

For a commutative quantale, the implication operator \rightarrow , also known as residuum, is given by: $\rightarrow: L \times L \rightarrow L, \alpha \rightarrow \beta = \bigvee \{ \gamma \in L \mid \alpha * \gamma \leq \beta \}$.

Definition 2.2. [34] A triangular norm or *t-norm* in short, is a triple $(L = [0, 1], \leq, *)$, where $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying conditions (GLM1), (GLM2) and that $\forall \alpha, \beta, \gamma \in [0, 1]: \alpha \leq \beta$ implies $\alpha * \gamma \leq \beta * \gamma$.

A *t-norm* is continuous if it is continuous as a mapping from $[0, 1] \times [0, 1]$ to $[0, 1]$. Among the important *t-norms*, we name a few, such as, minimum *t-norm* $\alpha * \beta = \alpha \wedge \beta$ and product *t-norm* $\alpha * \beta = \alpha \beta$.

Note that for a continuous *t-norm*, $([0, 1], \leq, *)$ is a **GL-monoid**.

Definition 2.3. [24, 27] A triple (L, \leq, \otimes) , where $\otimes: L \times L \rightarrow L$ is a binary operation on L , is called a **cl-premonoid** if and only if the following conditions are fulfilled:

(CP1) $\forall \alpha_1, \alpha_2, \beta_1, \beta_2 \in L: \alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$ implies $\alpha_1 \otimes \alpha_2 \leq \beta_1 \otimes \beta_2$;

(CP2) $\forall \alpha \in L: \alpha \leq \alpha \otimes \top$ and $\alpha \leq \top \otimes \alpha$;

(CP3) $\gamma \otimes (\bigvee_{k \in K} \alpha_k) = \bigvee_{k \in K} (\gamma \otimes \alpha_k)$, and

$(\bigvee_{k \in K} \alpha_k) \otimes \gamma = \bigvee_{k \in K} (\alpha_k \otimes \gamma)$ for $K \neq \emptyset, k \in K, \alpha_k, \gamma \in L$, is satisfied.

Definition 2.4. [24, 28] The quadruple $(L, \leq, *, \otimes)$ is called an **enriched cl-premonoid** if and only if the following are fulfilled:

(CLP1) $(L, \leq, *)$ is a **GL-monoid**;

(CLP2) (L, \leq, \otimes) is a **cl-premonoid**;

(CLP3) $*$ is dominated by \otimes : $\forall \alpha, \beta, \gamma, \delta \in L, (\alpha \otimes \beta) * (\gamma \otimes \delta) \leq (\alpha * \gamma) \otimes (\beta * \delta)$.

An enriched *cl-premonoid* $L = (L, \leq, *, \otimes)$ is said to be **pseudo-bisymmetric** if it satisfies the following axiom: $(\alpha * \beta) \otimes (\gamma * \delta) = ((\alpha \otimes \gamma) * (\beta \otimes \delta)) \vee ((\alpha \otimes \perp) * (\beta \otimes \top)) \vee ((\perp \otimes \gamma) * (\top \otimes \delta)), \forall \alpha, \beta, \gamma, \delta \in L$.

Remark 2.5. [28] (a) If $\otimes = *$, then the quadruple $(L, \leq, *, \otimes = *)$, is a **pseudo-bisymmetric enriched cl-premonoid**.

(b) If $*$ = \wedge and $\otimes = \wedge$, then the quadruple $(L, \leq, \wedge, \wedge)$ is a **frame**, which is a special case of (a).

Proposition 2.6. [27, 28] Let $(L, \leq, *)$ be a **GL-monoid**. Then the following are fulfilled $\forall \alpha, \beta, \gamma, \delta, \beta_j \in L$:

(1) $\alpha \leq \beta \rightarrow \gamma \Leftrightarrow \alpha * \beta \leq \gamma$;

(2) $\alpha * (\alpha \rightarrow \beta) \leq \beta$;

(3) $\alpha \leq \beta \Rightarrow \alpha \rightarrow \gamma \geq \beta \rightarrow \gamma$;

(4) $\alpha \leq \beta \Rightarrow \gamma \rightarrow \alpha \leq \gamma \rightarrow \beta$;

(5) $(\alpha \rightarrow \beta) \rightarrow \beta \geq \alpha$;

(6) $\alpha * (\beta \rightarrow \gamma) \leq \beta \rightarrow (\alpha * \gamma)$;

(7) $\alpha \rightarrow (\bigwedge_{j \in J} \beta_j) = \bigwedge_{j \in J} (\alpha \rightarrow \beta_j)$;

(8) $(\alpha \rightarrow \gamma) * (\beta \rightarrow \delta) \leq \alpha * \beta \rightarrow \gamma * \delta$;

(9) $\alpha \leq \beta \Leftrightarrow \alpha \rightarrow \beta = \top$;

(10) $\alpha \rightarrow \top = \top, \top \rightarrow \alpha = \alpha$, and $\perp \rightarrow \alpha = \top$.

In what follows, the quadruple $L = (L, \leq, *, \wedge)$ (or simply L) is assumed to be an enriched cl -premonoid, unless otherwise specified.

If $\alpha \in L$ and $A \subseteq X$, then the map $\alpha_A : X \rightarrow L$ is defined by

$$\alpha_A(x) = \begin{cases} \alpha, & \text{if } x \in A; \\ \perp, & \text{otherwise.} \end{cases}$$

In particular, $\top_X(x) = \top$, the characteristic function of X and $\perp_X(x) = \perp$, the zero function.

If $A := \{x\}$, then the characteristic function of the singleton is denoted by $\top_x := x$.

The set of all L -sets is denoted by $L^X (= \{v : X \rightarrow L\})$.

If \cdot is a binary operation on a set X , then \odot is a binary operation on L^X . Thus, for any $v_1, v_2 \in L^X$ and $z \in X$, $v_1 \odot v_2 : X \rightarrow L$ is defined by: $v_1 \odot v_2(z) = \bigvee \{v_1(x) * v_2(y) \mid x, y \in X, x \cdot y = z\}$; sometime, we write xy instead of $x \cdot y$. In particular, if (X, \cdot) is a group and $x \in X$, then for any $v \in L^X$ and $z \in X$, $x \odot v(z) = \bigvee_{ab=zx} \top_x(a) * v(b) = \bigvee_{xb=zx} \top * v(b) = v(x^{-1}z)$ (since \top is the unit element of L by Definition 2.1(GLM2)). Throughout the text we assume e as the identity element of the group (X, \cdot) .

If $v_1, v_2 \in L^X$, and $\rightarrow, *, \otimes$ are operations on L as explained before, then these operations are carried over to L^X point-wise:

- (i) $(v_1 \rightarrow v_2)(x) = v_1(x) \rightarrow v_2(x)$;
- (ii) $(v_1 * v_2)(x) = v_1(x) * v_2(x)$;
- (iii) $(v_1 \otimes v_2)(x) = v_1(x) \otimes v_2(x), \forall x \in X$.

Definition 2.7. [28] A map $\mathcal{F} : L^X \rightarrow L$ is called an L -filter on X if and only if the conditions below are satisfied:

(LF1) $\mathcal{F}(\top_X) = \top, \mathcal{F}(\perp_X) = \perp$;

(LF2) if $v_1, v_2 \in L^X$ with $v_1 \leq v_2$, then $\mathcal{F}(v_1) \leq \mathcal{F}(v_2)$;

(LF3) $\mathcal{F}(v_1) \otimes \mathcal{F}(v_2) \leq \mathcal{F}(v_1 \otimes v_2), \forall v_1, v_2 \in L^X$.

(SL) An L -filter \mathcal{F} is called a stratified L -filter if $\forall \alpha \in L, \forall \mu \in L^X, \alpha * \mathcal{F}(\mu) \leq \mathcal{F}(\alpha * \mu)$.

The set of all ordinary filters is denoted by $\mathbb{F}(X)$, and the set of all stratified L -filters on X is denoted by $\mathcal{F}_L^s(X)$. On $\mathcal{F}_L^s(X)$, partial ordering \leq is defined by: if $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, then $\mathcal{F} \leq \mathcal{G} \Leftrightarrow \mathcal{F}(v) \leq \mathcal{G}(v), \forall v \in L^X$. If $x \in X$, then $[x] \in \mathcal{F}_L^s(X)$, called point stratified L -filter on X , and is defined as $[x](v) = v(x)$, for all $v \in L^X$.

If $f : X \rightarrow Y$ is a function, then $f^\leftarrow : L^Y \rightarrow L^X$ is defined for any $\mu \in L^Y$ by $f^\leftarrow(\mu) = \mu \circ f$; and $f^\rightarrow : L^X \rightarrow L^Y$ is defined by $f^\rightarrow(v)(y) = \bigvee \{v(x) \mid f(x) = y\}$, for all $v \in L^X, y \in Y$.

Moreover, if $\mathcal{F} \in \mathcal{F}_L^s(X)$, then the stratified L -filter $f^\Rightarrow(\mathcal{F}) : L^Y \rightarrow L$ on Y is defined for any $\mu \in L^Y$ by $[f^\Rightarrow(\mathcal{F})](\mu) = \mathcal{F}(f^\leftarrow(\mu)) = \mathcal{F}(\mu \circ f)$.

If $\mathcal{F} \in \mathcal{F}_L^s(Y)$, then $f^\Leftarrow(\mathcal{F}) : L^X \rightarrow L$ is defined by $[f^\Leftarrow(\mathcal{F})](v) = \bigvee \{\mathcal{F}(\mu) \mid \mu \in L^Y, f^\leftarrow(\mu) \leq v\}$, for all $v \in L^X$, is a stratified L -filter on X if and only if for all $\mu \in L^Y, f^\leftarrow(\mu) = \perp_X \Rightarrow \mathcal{F}(\mu) = \perp$.

If $v \in L^X$ and $\mu \in L^Y$, then the product $v \times \mu \in L^{X \times Y}$ is defined by $v \times \mu = v \circ pr_1 * \mu \circ pr_2$, where $pr_1 : X \times Y \rightarrow X, (x, y) \mapsto x$ and $pr_2 : X \times Y \rightarrow Y, (x, y) \mapsto y$ are usual projections. Note that operation above on $*$ holds only for finite cases; otherwise, we need to take $*$ = \wedge . However, for our case, this does not create any problem with $*$ as we work for finite case.

Proposition 2.8. [24] If $(L, \leq, *, \otimes = *)$ is an enriched cl -premonoid, then for stratified L -filters \mathcal{F}_1 and \mathcal{F}_2 , the supremum $\mathcal{F}_1 \vee \mathcal{F}_2$ exists if and only if $\mathcal{F}_1(v_1) * \mathcal{F}_2(v_2) = \perp \forall v_1, v_2 \in L^X$ such that $v_1 * v_2 = \perp_X$. In particular, the supremum is the stratified L -filter defined for all $v \in L^X$ by $\mathcal{F}_1 \vee \mathcal{F}_2(v) = \bigvee \{\mathcal{F}_1(v_1) * \mathcal{F}_2(v_2) \mid v_1, v_2 \in L^X, v_1 * v_2 \leq v\}$.

Remark 2.9. [16](see also, [30]) If $(L, \leq, *, \otimes)$ is a pseudo-bisymmetric enriched cl -premonoid, then in view of the Definition 3.5[16], for $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $\mathcal{G} \in \mathcal{F}_L^s(Y)$, their product $\mathcal{F} \times \mathcal{G} \in \mathcal{F}_L^s(X \times Y)$, where the product is given by

$$\mathcal{F} \times \mathcal{G} = pr_1^\Leftarrow(\mathcal{F}) \vee pr_2^\Leftarrow(\mathcal{G}).$$

In particular, for a frame L , if $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $\mathcal{G} \in \mathcal{F}_L^s(Y)$ and $v \in L^{X \times Y}$, we have

$$\mathcal{F} \times \mathcal{G}(v) = \bigvee \{\mathcal{F}(v_1) \wedge \mathcal{G}(v_2) : v_1 \times v_2 \leq v\}.$$

If $L = (L, \leq, *, \otimes)$ is a pseudo-bisymmetric enriched cl -premonoid, (X, \cdot) is a group and $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, then the map $\mathcal{F} \odot \mathcal{G}: L^X \rightarrow L$ is defined for any $v \in L^X$ by: $\mathcal{F} \odot \mathcal{G}(v) = \bigvee \{\mathcal{F}(v_1) * \mathcal{G}(v_2) \mid v_1, v_2 \in L^X, v_1 \odot v_2 \leq v\}$. If (X, \cdot) is a group and $\mathcal{F} \in \mathcal{F}_L^s(X)$, then \mathcal{F}^{-1} is defined by $\mathcal{F}^{-1}(v) = \mathcal{F}(v^{-1})$, where $v^{-1}: X \rightarrow L, x \mapsto v(x^{-1})$. Clearly, $\mathcal{F}^{-1} \in \mathcal{F}_L^s(X)$, since for any $v \in L^X, j^{\rightarrow}(\mathcal{F})(v) = \mathcal{F}(j^{\leftarrow}(v)) = \mathcal{F}(v^{-1}) = \mathcal{F}^{-1}(v)$, where $j: X \rightarrow X, x \mapsto x^{-1}$. Also, if $m: X \times X \rightarrow X, (x, y) \mapsto xy$, then for any $v_1, v_2 \in L^X$ and $z \in X, m^{\rightarrow}(v_1 \times v_2)(z) = \bigvee_{m(x,y)=z} (v_1 \times v_2)(x, y) = \bigvee_{xy=z} (v_1 \circ pr_1 * v_2 \circ pr_2)(x, y) = \bigvee_{xy=z} v_1 \circ pr_1(x, y) * v_2 \circ pr_2(x, y) = \bigvee_{xy=z} v_1(x) * v_2(y) = v_1 \odot v_2(z)$. Furthermore, for a group (G, \cdot) , if $\mathbb{F}, \mathbb{G} \in \mathbb{F}(G)$, then $\mathbb{F} \odot \mathbb{G}$ is defined as a filter generated by the sets $F \cdot G = \{pq \mid p \in F, q \in G\}$, where $F \in \mathbb{F}$ and $G \in \mathbb{G}$; usually, we denote it by $\mathbb{F} \odot \mathbb{G} = [\{F \cdot G \mid F \in \mathbb{F}, G \in \mathbb{G}\}]$.

Lemma 2.10. [3] Let $L = (L, \leq, *, \otimes = *)$ be a GL -monoid and (X, \cdot) be a group. Then for any $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $m^{\rightarrow}(\mathcal{F} \times \mathcal{G}) = \mathcal{F} \odot \mathcal{G}$.

Note that if $L = ([0, 1], \leq, *, \otimes = \wedge)$ with a t -norm $*$, then the above lemma is always true. But if $L = (L, \leq, *, \otimes)$, then since we do not have an explicit formula for the product L -filter, we cannot say in the perspective of the preceding lemma that this lemma holds.

3. Continuous action of enriched lattice-valued convergence groups on enriched lattice-valued convergence spaces

Definition 3.1. [30, 41] Let $L = (L, \leq, *, \otimes = \wedge)$ be an enriched cl -premonoid and $\lim: \mathcal{F}_L^s(X) \rightarrow L^X$ be a mapping such that the following are satisfied:

- (C1) $\forall x \in X, \limx = \top$;
- (C2) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ with $\mathcal{F} \leq \mathcal{G}$, and $\forall x \in X, \lim \mathcal{F}(x) \leq \lim \mathcal{G}(x)$;
- (C3) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X), \forall x \in X, \lim \mathcal{F}(x) * \lim \mathcal{G}(x) \leq \lim(\mathcal{F} \wedge \mathcal{G})(x)$;
- (C3s) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X), \forall x \in X, \lim \mathcal{F}(x) \wedge \lim \mathcal{G}(x) \leq \lim(\mathcal{F} \wedge \mathcal{G})(x)$.

Then the pair (X, \lim) is called an stratified enriched cl -premonoid-valued convergence space (or simply by enriched latticed-valued convergence space if conditions (C1)-(C3) hold, if the conditions (C1),(C2) and (C3s) hold, then we speak of stratified enriched cl -premonoid-valued strong convergence space or simply by enriched strong lattice-valued convergence space.

A mapping $f: (X, \lim) \rightarrow (X', \lim')$ between enriched lattice-valued convergence spaces (resp. between enriched lattice-valued strong convergence spaces) is called continuous if and only if $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X, \lim \mathcal{F}(x) \leq \lim' f^{\rightarrow}(\mathcal{F})(f(x))$.

The category of all stratified enriched cl -premonoid-valued convergence spaces and continuous mappings is denoted by $SL\text{-CONV}$ (resp. the category of all stratified enriched cl -premonoid-valued strong convergence spaces is denoted by $SSL\text{-CONV}$).

Proposition 3.2. [16, 41] (see also [30]) The category $SL\text{-CONV}$ (resp. $SSL\text{-CONV}$) is topological over the category SET . In particular, if $(f_j: X \rightarrow (Y_j, \lim_j))_{j \in J}$ is a source, then the initial structure $\lim: \mathcal{F}_L^s(X) \rightarrow L^X$ on X is given for any $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X$ by

$$\lim \mathcal{F}(x) = \bigwedge_{j \in J} \lim_j f_j^{\rightarrow}(\mathcal{F})(f_j(x)).$$

For the convenience of the reader, we however, recall the product structure of lattice-valued convergence spaces as given in [16, 41] (see [30] for the case of frame), although we will be using this for some specific lattices.

Let $L = (L, \leq, *, \otimes)$ be a pseudo-bisymmetric enriched cl -premonoid. Let $(X, \lim), (Y, \overline{\lim})$ be SL -convergence spaces, then their product $\lim \times \overline{\lim}: \mathcal{F}_L^s(X \times Y) \rightarrow L^{X \times Y}$ defined for an $\mathcal{F} \in \mathcal{F}_L^s(X \times Y)$ by

$$(\lim \times \overline{\lim}) \mathcal{F} = pr_1^{\leftarrow}(\lim pr_1^{\rightarrow}(\mathcal{F})) \wedge pr_2^{\leftarrow}(\overline{\lim} pr_2^{\rightarrow}(\mathcal{F})).$$

The pair $(X \times Y, \lim \times \overline{\lim})$ is a product stratified L -convergence space.

Definition 3.3. [2, 3] Let $L = (L, \leq, *, \wedge)$ be an enriched cl-premonoid, $(G, \cdot) \in |\mathbf{GRP}|$, and $(G, \lim) \in |\mathbf{SL-CONV}|$ (resp. $(G, \lim) \in |\mathbf{SSL-CONV}|$). Then the triple (G, \cdot, \lim) is called a stratified L-convergence group (resp. stratified strong L-convergence group) if and only if the following axioms are satisfied:

(CGM) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(G), \forall x, y \in G: \lim \mathcal{F}(x) * \lim \mathcal{G}(y) \leq \lim(\mathcal{F} \odot \mathcal{G})(xy)$ (resp. $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y) \leq \lim(\mathcal{F} \odot \mathcal{G})(xy)$), whenever $\mathcal{F} \odot \mathcal{G}$ is stratified L-filter;
 (CGI) $\forall \mathcal{F} \in \mathcal{F}_L^s(G) \forall x \in X, \lim \mathcal{F}(x) \leq \lim \mathcal{F}^{-1}(x^{-1})$.

The category of all stratified L-convergence groups and continuous group homomorphisms is denoted by **SL-CONVGRP**. Similarly, the category **SSL-CONVGRP** is described.

Remark 3.4. [2] If $(L, \leq, *, \otimes) = (L, \leq, * = \wedge, \otimes = \wedge)$, i.e., when L is a frame, then (CGM) is equivalent to the continuity of multiplication $m: G \times G \rightarrow G, (x, y) \mapsto xy$. In fact, viewing product stratified strong L-convergence structure for frame L as in the case of [30], one can say that a mapping $m: (G, \lim) \times (G, \lim) \rightarrow (G, \lim), (x, y) \mapsto xy$ is continuous if and only if for all stratified L-filters \mathcal{F}, \mathcal{G} , and for all $(x, y), \lim pr_1^{\rightarrow}(\mathcal{F} \times \mathcal{G})(x) \wedge \lim pr_2^{\rightarrow}(\mathcal{F} \times \mathcal{G})(y) \leq \lim m^{\rightarrow}(\mathcal{F} \times \mathcal{G})(m(x, y))$ which in turn yields that $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y) = \lim pr_1^{\rightarrow}(\mathcal{F} \times \mathcal{G})(x) \wedge \lim pr_2^{\rightarrow}(\mathcal{F} \times \mathcal{G})(y) \leq \lim m^{\rightarrow}(\mathcal{F} \times \mathcal{G})(m(x, y))$. In view of Lemma 2.10, this amounts to say that $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y) \leq \lim(\mathcal{F} \odot \mathcal{G})(xy)$.

Definition 3.5. Let $L = (L, \leq, *, \otimes = \wedge)$ be an enriched cl-premonoid. Let $(G, \cdot, \lim_G) \in |\mathbf{SL-CONVGRP}|$ (resp. $(G, \cdot, \lim_G) \in |\mathbf{SSL-CONVGRP}|$), and $(X, \lim_X) \in |\mathbf{SL-CONV}|$ (resp. $(X, \lim_X) \in |\mathbf{SSL-CONV}|$). Then the triple $((G, \cdot, \lim_G), (X, \lim_X), \varphi)$ is called a stratified L-convergence transformation group (resp. stratified strong L-convergence transformation group) on a stratified (resp. stratified strong) L-convergence space with respect to $\varphi: G \times X \rightarrow X, (g, x) \mapsto \varphi(g, x)$ if the following axioms are satisfied:

(CTG1) $\forall \mathcal{H} \in \mathcal{F}_L^s(G), \forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall g \in G$ and $x \in X$:

$$\begin{aligned} \lim_G \mathcal{H}(g) * \lim_X \mathcal{F}(x) &\leq \lim_X (\varphi^{\rightarrow}(\mathcal{H} \times \mathcal{F}))(\varphi(g, x)) \\ (\text{resp. } \lim_G \mathcal{H}(g) \wedge \lim_X \mathcal{F}(x) &\leq \lim_X (\varphi^{\rightarrow}(\mathcal{H} \times \mathcal{F}))(\varphi(g, x))); \end{aligned}$$

(CTG2) $\varphi(e, x) = x, \forall x \in X$;

(CTG3) $\varphi(g \cdot h, x) = \varphi(g, \varphi(h, x)), \forall g, h \in G$ and $\forall x \in X$.

Here the function φ is called continuous action of G on X , the enriched cl-premonoid-valued stratified convergence group or in short, stratified L-convergence group (G, \cdot, \lim_G) is called a phase group, and the stratified L-convergence space (X, \lim_X) is called phase space.

The category of stratified L-convergence transformation groups (resp. stratified strong L-convergence transformation groups) denoted by **SL-CONVTGRP** (resp. **SSL-CONVTGRP**) consists of all stratified L-convergence transformation groups (resp. stratified strong L-convergence transformation groups) as objects, and all pairs $(k, f): (G, X, \varphi) \rightarrow (G', X', \varphi')$ as morphism, where

(TG1) $G \xrightarrow{k} G'$ is a **SL-CONVGRP**-morphism (resp. **SSL-CONVGRP**-morphism), i.e., continuous group homomorphism;

(TG2) $X \xrightarrow{f} X'$ is a **SL-CONV**-morphism (resp. **SSL-CONV**-morphism), i.e., a continuous mapping such that $\varphi' \circ (k \times f) = f \circ \varphi$.

Remark 3.6. We would like to point out briefly that in the introduction we already mentioned that the notion of lattice-valued convergence is an extension of $\{0, 1\}$ -valued convergence which can be identified with classical convergence as mentioned in a remark in [30]. Considering this point, and exploiting the relationship between classical filters and stratified L-filters (see f.i. Section 6 [41] and Section 3 [31], see also [24]), one can view that our present study of lattice-valued convergence transformation group is an extension of classical convergence transformation group.

Example 3.7. Any stratified L-convergence groups (G, \cdot, \lim) can be made into a stratified L-convergence transformation group $((G, \cdot, \lim), (G, \lim), \varphi)$ on itself in the following way: $((G, \cdot, \lim), (G, \lim), \varphi)$, where $\varphi: G \times G \rightarrow G$ defined by $\varphi(t, s) = ts$, the left translation in G .

Example 3.8. Let $((G, \cdot, \lim_G), (X, \lim_X), \varphi) \in |\mathbf{SL-CONVTGRP}|$ on a stratified L-convergence group (X, \lim_X) with respect to φ . Then $((G, \cdot, \lim_G), (X \times X, \lim_X \times \lim_X), \varphi')$, where $\varphi': G \times (X \times X) \rightarrow (X \times X)$ is defined by

$\varphi'(g, (x, x')) = (\varphi(g, x), \varphi(g, x'))$, for all $g \in G$ and $(x, x') \in X \times X$, is a stratified L -convergence transformation group on the product stratified L -convergence space.

Example 3.9. Let $((G, \cdot, \lim), (X, \lim), \varphi) \in |\text{SL-CONVTGRP}|$ on a stratified L -convergence space (X, \lim) with respect to φ , and let $((G, \cdot, \lim), (Y, \lim'), \psi) \in |\text{SL-CONVTGRP}|$ on a stratified L -convergence space (Y, \lim') with respect to ψ . Then $((G, \cdot, \lim), (X \times Y, \lim \times \lim'), \gamma) \in |\text{SL-CONVTGRP}|$ on a stratified L -convergence space $(X \times Y, \lim \times \lim')$ with respect to $\gamma: G \times (X \times Y) \rightarrow (X \times Y)$, is defined by $\gamma(g, (x, y)) = (\varphi(g, x), \psi(g, y))$. In fact, for $g, h \in G$, and $(x, y) \in X \times Y$,

$$\gamma(g \cdot h, (x, y)) = (\varphi(g \cdot h, x), \psi(g \cdot h, y)) = (\varphi(g, \varphi(h, x)), \psi(g, \psi(h, y))) = \gamma(g, (\varphi(h, x), \psi(h, y))) = \gamma(g, \gamma(h, (x, y))).$$

The continuity of γ follows from the continuity of φ and ψ .

Definition 3.10. [30] Let (L, \leq, \wedge) be a frame, $(X, \lim), (Y, \lim') \in |\text{SSL-CONV}|$ and $C(X, Y) = \{f: (X, \lim) \rightarrow (Y, \lim') : f \text{ is continuous}\}$. If $c\text{-lim}: \mathcal{F}_L^s(C(X, Y)) \rightarrow L^{C(X, Y)}$ is defined for any $\Phi \in \mathcal{F}_L^s(C(X, Y))$ and $f \in C(X, Y)$ by

$$c\text{-lim } \Phi(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim ev^\Rightarrow (\Phi \times \mathcal{F})(f(x))],$$

where $ev: C(X, Y) \times X \rightarrow Y, (f, x) \mapsto f(x)$ is called an evaluation mapping whence limit $c\text{-lim}$ is called stratified L -convergence of continuous convergence.

Lemma 3.11. [30, 33] Let $f: X \rightarrow Y$ be a mapping and $ev: C(X, Y) \times X \rightarrow Y$ be the evaluation mapping. Then for all $\mathcal{F} \in \mathcal{F}_L^s(X)$, $f^\Rightarrow(\mathcal{F}) = ev^\Rightarrow([\mathcal{F}] \times \mathcal{F})$.

Lemma 3.12. [30] If L is a frame, then $(C(X, Y), c\text{-lim}) \in |\text{SSL-CONV}|$.

Definition 3.13. Let $(X, \lim) \in |\text{SSL-CONV}|$ and $\mathcal{H}(X) = \{f: X \rightarrow X : f \text{ is a homeomorphism, i.e., a bijective and both } f, f^{-1} \text{ are continuous}\}$. Define a stratified strong L -convergence structure $c\text{-lim}: \mathcal{F}(\mathcal{H}(X)) \rightarrow L^{\mathcal{H}(X)}$ on $\mathcal{H}(X)$ the group of homeomorphisms under composition: for any $\Phi \in \mathcal{F}_L^s(\mathcal{H}(X))$ and $f \in \mathcal{H}(X)$ by

$$c\text{-lim } (\Phi)(f) = c\text{-lim } (\Phi)(f) \wedge c\text{-lim}^{(-1)} (\Phi)(f), \text{ where}$$

$$c\text{-lim } (\Phi)(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim ev^\Rightarrow (\Phi \times \mathcal{F})(f(x))] \quad (3.1),$$

and

$$c\text{-lim}^{(-1)} (\Phi)(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim ev^\Rightarrow (\Phi^{-1} \times \mathcal{F})(f^{-1}(x))] \quad (3.2).$$

Remark 3.14. Note that in view of the Proposition 2.6, and upon combining preceding items (3.1) and (3.2), one can write the above definition as follows:

$$c\text{-lim } (\Phi)(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} \lim \mathcal{F}(x) \rightarrow [\lim ev^\Rightarrow (\Phi \times \mathcal{F})(f(x)) \wedge \lim ev^\Rightarrow (\Phi^{-1} \times \mathcal{F})(f^{-1}(x))], \text{ where } \Phi^{-1} = [\{\phi^{-1}: \phi \in \Phi\}, \phi^{-1} = \{ \{f^{-1} \in \mathcal{H}(X): f \in \mathcal{H}(X)\}] .$$

Theorem 3.15. Let $L = (L, \leq, \wedge)$ be a frame, and \lim be a stratified strong L -convergence structure on X , then $(\mathcal{H}(X), \cdot, c\text{-lim}) \in |\text{SSL-CONVGRP}|$, where $c\text{-lim} = c\text{-lim} \wedge c\text{-lim}^{(-1)}$.

Proof. (C1) follows upon using Lemma 3.11 ([30]: Lemma 8.2), while (C2) follows from Proposition 8.3 [30]. (C3s) Upon using exactly the similar rout as in Lemma 4.3 [32], one can proof this part. We prove only (CGM) and (CGI):

(CGM) Let $\Phi, \Psi \in \mathcal{F}_L^s(\mathcal{H}(X))$ and $f, g \in \mathcal{H}(X)$. We show that

$$c\text{-lim } \Phi(f) \wedge c\text{-lim } \Psi(g) \leq c\text{-lim}(\Phi \circ \Psi)(f \cdot g).$$

We have $c\text{-lim } \Phi(f) \wedge (c\text{-lim } \Psi(g) \wedge \mathcal{F}(x))$

$$\leq c\text{-lim } \Phi(f) \wedge \lim ev^\Rightarrow (\Psi \times \mathcal{F})(g(x))$$

$$\leq \lim ev^\Rightarrow (\Phi \times ev^\Rightarrow (\Psi \times \mathcal{F}))(f \cdot g(x)) = \lim ev^\Rightarrow ((\Phi \circ \Psi) \times \mathcal{F})(f \cdot g(x)).$$

We give here for the reader an explicit proof upon using evaluation mapping for the validity of the inequality $ev^\Rightarrow (\Phi \times ev^\Rightarrow (\Psi \times \mathcal{F})) \leq ev^\Rightarrow ((\Phi \circ \Psi) \times \mathcal{F})$ for better understanding. For that take $v \in L^{\mathcal{H}(X) \times (\mathcal{H}(X) \times X)}$, then

for any $f, g \in \mathcal{H}(X)$,

$$\begin{aligned} ev^{\Rightarrow}(\Phi \times ev^{\Rightarrow}(\Psi \times \mathcal{F}))(v) &= [\Phi \times ev^{\Rightarrow}(\Psi \times \mathcal{F})](ev^{\leftarrow}(v)) \\ &= \bigvee_{v_1 \times v_2 \leq ev^{\leftarrow}(v), v_1 \in L^{\mathcal{H}(X)}, v_2 \in L^{\mathcal{H}(X) \times X}} [\Phi(v_1) \wedge (\bigvee_{\rho_1 \times \rho_2 \leq ev^{\leftarrow}(v_2), \rho_1 \in L^{\mathcal{H}(X)}, \rho_2 \in L^X} \Psi(\rho_1) \wedge \mathcal{F}(\rho_2))] \\ &= \bigvee_{v_1 \times v_2 \leq ev^{\leftarrow}(v); \rho_1 \times \rho_2 \leq ev^{\leftarrow}(v_2)} (\Phi(v_1) \wedge \Psi(\rho_1)) \wedge \mathcal{F}(\rho_2) \\ &\leq \bigvee_{v_1(f) \wedge v_2(ev(g, x)) \leq v(ev(f, ev(g, x))); \rho_1(g) \wedge \rho_2(x) \leq v_2(ev(g, x))} (\Phi(v_1) \wedge \Psi(\rho_1)) \wedge \mathcal{F}(\rho_2) \\ &\leq \bigvee_{v_1(f) \wedge \rho_1(g) \wedge \rho_2(x) \leq ev^{\leftarrow}(v)(f, g, x)} (\Phi(v_1) \wedge \Psi(\rho_1)) \wedge \mathcal{F}(\rho_2) \\ &\leq \bigvee_{v_1(f) \wedge \rho_1(g) \wedge \rho_2(x) \leq ev^{\leftarrow}(v)(f, g, x)} (\bigvee_{v_1 \circ \rho_1 \leq \xi} (\Phi \odot \Psi)(\xi)) \wedge \mathcal{F}(\rho_2) \\ &= \bigvee_{v_1(f) \wedge \rho_1(g) \wedge \rho_2(x) \leq ev^{\leftarrow}(v)(f, g, x); v_1(f) \wedge \rho_1(g) \leq \xi(f, g)} (\Phi \odot \Psi)(\xi) \wedge \mathcal{F}(\rho_2) \\ &\leq \bigvee_{\xi(f, g) \wedge \rho_2(x) \leq \varphi^{\leftarrow}(v)(f, g, x)} (\Phi \odot \Psi)(\xi) \wedge \mathcal{F}(\rho_2) \\ &= \bigvee_{\xi \times \rho_2 \leq \varphi^{\leftarrow}(v)} (\Phi \odot \Psi)(\xi) \wedge \mathcal{F}(\rho_2) \\ &= ev^{\Rightarrow}(\Phi \odot \Psi \times \mathcal{F})(v), \text{ i.e., } ev^{\Rightarrow}(\Phi \times ev^{\Rightarrow}(\Psi \times \mathcal{F})) \leq ev^{\Rightarrow}(\Phi \odot \Psi \times \mathcal{F}). \end{aligned}$$

Thus, we have

$$\begin{aligned} c - \lim \Phi(f) \wedge c - \lim \Psi(g) &\leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim ev^{\Rightarrow}((\Phi \odot \Psi) \times \mathcal{F})](f \cdot g(x)), \text{ i.e.,} \\ c - \lim \Phi(f) \wedge c - \lim \Psi(g) &\leq c - \lim \Phi \odot \Psi(f \cdot g). \end{aligned}$$

Similarly, one can show that $c - \lim^{(-1)} \Phi(f) \wedge c - \lim^{(-1)} \Psi(g) \leq c - \lim^{(-1)} \Phi \odot \Psi(f \cdot g)$.

(CGI) Let $\Phi \in \mathcal{F}_L^s(\mathcal{H}(X))$ and $f \in \mathcal{H}(X)$. Then we have from Definition 3.13 (3.1),

$$c - \lim \Phi(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim ev^{\Rightarrow}(\Phi \times \mathcal{F})(f(x))] \quad (3.3)$$

We apply the continuities of $j: x \rightarrow x^{-1}$ and the composition $j \circ ev: \mathcal{H}(X) \times X \rightarrow X$ together with the fact that $(ev^{\Rightarrow}(\Phi \times \mathcal{F}))^{-1} = ev^{\Rightarrow}(\Phi^{-1} \times \mathcal{F})$. In fact,

$$\begin{aligned} \text{for any } v \in L^{\mathcal{H}(X) \times X}, (ev^{\Rightarrow}(\Phi \times \mathcal{F}))^{-1}(v) &= ev^{\Rightarrow}(\Phi \times \mathcal{F})(v^{-1}) = \Phi \times \mathcal{F}(ev^{\leftarrow}(v^{-1})) \\ &= \bigvee_{v_1 \times v_2 \leq ev^{\leftarrow}(v^{-1}), v_1 \in L^{\mathcal{H}(X)}, v_2 \in L^X} \Phi(v_1) \wedge \mathcal{F}(v_2) \quad (3.4). \end{aligned}$$

Now for any $f \in \mathcal{H}(x)$ and $x \in X$, we have

$$\begin{aligned} v_1(f) \wedge v_2(x) &\leq v^{-1}(ev(f, x)) = v^{-1}(f(x)) = v((f(x))^{-1}) \\ &= v(f^{-1}(x)) = (f^{-1})^{\leftarrow}(v)(x) \implies v_1^{-1}(f^{-1}) \wedge v_2(x) \leq (f^{-1})^{\leftarrow}(v)(x) \\ &\implies (v_1^{-1} \times v_2)(f^{-1}, x) \leq ev^{\leftarrow}(v)(f^{-1}, x), \text{ i.e., } v_1^{-1} \times v_2 \leq ev^{\leftarrow}(v). \end{aligned}$$

Then continuing after (3.4), we have

$$\begin{aligned} (ev^{\Rightarrow}(\Phi \times \mathcal{F}))^{-1}(v) &= \bigvee_{v_1 \times v_2 \leq ev^{\leftarrow}(v^{-1}), v_1 \in L^{\mathcal{H}(X)}, v_2 \in L^X} \Phi(v_1) \wedge \mathcal{F}(v_2) \\ &\leq \bigvee_{v_1^{-1} \times v_2 \leq ev^{\leftarrow}(v)} (\Phi^{-1}(v_1^{-1}) \wedge \mathcal{F}(v_2)) = ev^{\Rightarrow}(\Phi^{-1} \times \mathcal{F})(v), \text{ i.e.,} \\ (ev^{\Rightarrow}(\Phi \times \mathcal{F}))^{-1} &\leq ev^{\Rightarrow}(\Phi^{-1} \times \mathcal{F}). \end{aligned}$$

Therefore, it follows from (3.3) that

$$c - \lim \Phi(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim ev^{\Rightarrow}(\Phi \times \mathcal{F})(f(x))]$$

$$\leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim ev^{\Rightarrow}(\Phi^{-1} \times \mathcal{F})(f^{-1}(x))]$$

$$= c - \lim \Phi^{-1}(f^{-1}). \text{ Similarly, one can show that } c - \lim^{(-1)} \Phi(f) \leq c - \lim^{(-1)} \Phi^{-1}(f^{-1})$$

which after combing together imply that $c - \lim \Phi(f) \leq c - \lim \Phi^{-1}(f^{-1})$. \square

Lemma 3.16. Let (L, \leq, \wedge) be a frame. Then the stratified strong L -convergence group $(\mathcal{H}(X), \cdot, c - \lim)$ has the property that for any stratified strong L -convergence space (Y, \cdot, \lim_Y) , the continuity of the mapping $\zeta: (Y, \lim_Y) \rightarrow (\mathcal{H}(X), c - \lim)$ implies the continuity of $\bar{\zeta}: Y \times X \rightarrow X$, where $\bar{\zeta}(y, x) = \zeta(y)(x)$, $(\zeta(y) \in \mathcal{H}(X), \text{ and } x \in X)$

Proof. Since ζ and the identity mappings id_X are continuous, $\zeta \times id_X$ is continuous, it follows then in conjunction with the continuity of ev , the composition $ev \circ \zeta \times id_X$ is continuous as shown in the commutative diagram below. Hence $\bar{\zeta}$ is continuous.

$$\begin{array}{ccc} Y \times X & \xrightarrow{\zeta \times id_X} & \mathcal{H}(X) \times X \\ & \searrow \bar{\zeta} & \downarrow ev \\ & & X \end{array}$$

\square

Now we present a characterization theorem.

Theorem 3.17. Let (L, \leq, \wedge) be a frame. Then the triple $((G, \cdot, \lim_G), (X, \lim_X), \varphi)$ is a stratified strong L -convergence transformation group on a stratified strong L -convergence space (X, \lim_X) with respect to the mapping $\varphi: G \times X \rightarrow X, (g, x) \mapsto \varphi(g, x)$, if and only if $\chi: (G, \lim_G) \rightarrow (\mathcal{H}(X), c - \lim)$ is a continuous homomorphism, where χ is defined by $\chi(g)(x) = \varphi(g, x), g \in G$, and $x \in X$.

Proof. Let $((G, \cdot, \lim_G), (X, \lim_X), \varphi) \in |\mathbf{SSL-CONVTGRP}|$. Define $\chi(g)(x) = \varphi(g, x)$ for $g \in G$ and $x \in X$. Then applying Definition 3.5(CTG3), we have for any $g' \in G, \chi_{gg'}(x') = \varphi(gg', x) = \varphi(g, \varphi(g', x)) = \varphi(g, \chi_{g'}(x)) = \chi_g(\chi_{g'}(x)) = (\chi_g \circ \chi_{g'})(x)$ implying $\chi_{gg'} = \chi_g \circ \chi_{g'}$. So, $\chi: G \rightarrow \mathcal{H}(X)$ is a homomorphism. To show that the mapping $\chi: (G, \lim_G) \rightarrow (\mathcal{H}(X), c - \lim)$ is continuous, we let $\mathcal{F} \in \mathcal{F}_L^s(G)$ and $g \in G$, and show that $\lim_G \mathcal{F}(g) \leq c - \lim \chi(\mathcal{F})(\chi_g)$. Now let $x \in X$ and $\mathcal{G} \in \mathcal{F}_L^s(X)$. Put $\chi^\Rightarrow(\mathcal{F}) = \Phi \in \mathcal{F}_L^s(\mathcal{H}(X))$. Since φ is continuous, we have

$\lim_G \mathcal{F}(g) \wedge \lim_X \mathcal{G}(x) \leq \lim_X \varphi^\Rightarrow(\Phi \times \mathcal{G})(\chi_g(x))$, whence $\chi_g(x) = \varphi(g, x)$.

Then one can write $\lim_G \mathcal{F}(g) \leq \lim_X \mathcal{G}(x) \rightarrow \lim_X ev^\Rightarrow(\Phi \times \mathcal{G})(\chi_g(x))$. This is true for any $\mathcal{G} \in \mathcal{F}_L^s(X)$ and for any $x \in X$ that

$$\begin{aligned} \lim_G \mathcal{F}(g) &\leq \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim_X \mathcal{G}(x) \rightarrow \lim_X ev^\Rightarrow(\Phi \times \mathcal{G})(\chi_g(x))] \\ &= c - \lim \Phi(\chi_g), \text{ i.e., } \lim_G(\mathcal{F})(g) \leq c - \lim \Phi(\chi_g) \quad (3.5). \end{aligned}$$

Next, as $(G, \cdot, \lim_G) \in |\mathbf{SSL-CONVGRP}|$, for any $\mathcal{F} \in \mathcal{F}_L^s(G), \lim_G \mathcal{F}(g) \leq \lim_G \mathcal{F}^{-1}(g^{-1})$, and since

$\chi: G \rightarrow \mathcal{H}(X)$ is a homomorphism $\chi_{g^{-1}} = (\chi_g)^{-1}$, we have

$$\lim_G \mathcal{F}(g) \leq \lim_X \mathcal{G}(x) \rightarrow \lim_X ev^\Rightarrow(\Phi^{-1} \times \mathcal{G})(\chi_g^{-1}(x))$$

implying that $\lim_G \mathcal{F}(g) \leq \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim_X \mathcal{G}(x) \rightarrow \lim ev^\Rightarrow(\Phi^{-1} \times \mathcal{G})(\chi_g^{-1}(x))]$.

This means that $\lim_G \mathcal{F}(g) \leq c - \lim^{(-1)} \Phi(\chi_g)$ (3.6).

Hence from (3.5) and (3.6), we have $\lim_G \mathcal{F}(g) \leq c - \lim \Phi(\chi_g) \wedge c - \lim \Phi^{-1}(\chi_g^{-1})$.

To show the converse, let $\chi: G \rightarrow \mathcal{H}(X)$ be continuous homomorphism, $g, h \in G$ and $x \in X$. Then $\varphi(gh, x) = \chi_{gh}(x) = \chi_g(\chi_h(x)) = \chi_g(\varphi(h, x)) = \varphi(g, \varphi(h, x))$ while the other condition is trivially true. The continuity of φ follows from the Lemma 3.16 with $Y = G$ and $\bar{\zeta} = \varphi$ (see also the diagram above).

Hence, the triple $((G, \cdot, \lim_G), (X, \lim_X), \varphi) \in |\mathbf{SSL-CONVTGRP}|$ over $(X, \lim_X) \in |\mathbf{SSL-CONV}|$ with respect to φ . \square

Theorem 3.18. Let (L, \leq, \wedge) be a frame, $(X, \cdot, \lim_X) \in |\mathbf{SSL-CONVGRP}|$ and $(X, \lim_X) \in |\mathbf{SSL-CONV}|$. If $\mathcal{H}(X)$ is a group of homeomorphisms of (X, \lim_X) , then X is isomorphically embedded in $(\mathcal{H}(X), c - \lim)$.

Proof. Let $\chi: X \rightarrow \mathcal{H}(X), x \mapsto \chi_x$ given by $\chi_x(x') = xx'$ for any $x' \in X$. The mapping χ is an injective-homomorphism, we show that the mapping χ is continuous, for, let $\mathcal{F} \in \mathcal{F}_L^s(X)$, we show that $\lim \mathcal{F}(x) \leq c - \lim \chi^\Rightarrow(\mathcal{F})(\chi_x)$.

Since (X, \cdot, \lim_X) is a stratified strong L -convergence group, for any $x' \in X$ and $\mathcal{G} \in \mathcal{F}_L^s(X)$, we have: $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(x') \leq \lim(\mathcal{F} \circ \mathcal{G})(xx') = \lim(\mathcal{F} \circ \lim \mathcal{G})(\chi_x(x'))$.

But it follows that $\mathcal{F} \circ \mathcal{G} \leq \chi^\Rightarrow(\mathcal{F})\mathcal{G}$, so

$$\begin{aligned} \lim \mathcal{F}(x) \wedge \lim \mathcal{G}(x') &\leq \lim ev^\Rightarrow(\chi^\Rightarrow(\mathcal{F}) \times \mathcal{G})(\chi_x(x')) \\ \Rightarrow \lim \mathcal{F}(x) &\leq \lim \mathcal{G}(x') \rightarrow \lim ev^\Rightarrow(\chi^\Rightarrow(\mathcal{F}) \times \mathcal{G})(\chi_x(x')) \end{aligned}$$

This is true for any $\mathcal{G} \in \mathcal{F}_L^s(X)$ and any $x' \in X$. This means that

$$\lim \mathcal{F}(x) \leq \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} \bigwedge_{x' \in X} [\lim \mathcal{G}(x') \rightarrow \lim ev^\Rightarrow(\chi^\Rightarrow(\mathcal{F}) \times \mathcal{G})(\chi_x(x'))]$$

$$= c - \lim \chi^\Rightarrow(\mathcal{F})(\chi_x). \text{ Similarly, } \lim \mathcal{F}(x) \leq c - \lim^{(-1)} \chi^\Rightarrow(\mathcal{F})(\chi_x)$$

Hence $\lim \mathcal{F}(x) \leq c - \lim \chi^\Rightarrow(\mathcal{F})(\chi_x)$.

Now let us show that the mapping $\chi_{\lim(X)}^{-1}: \chi(X) \rightarrow X$ is continuous. Pick $\chi_x \in \chi(X)$ and suppose $\Phi \in \mathcal{F}_L^s(\chi(X))$; we show that $c - \lim_{\lim(X)} \Phi(\chi_x) \leq \lim \chi^{-1}(\Phi)(\chi^{-1}(\chi_x))$.

Since $\chi_{\lim(X)}^{-1}(\chi_x) = x = xe = \chi_x(e) = ev(\chi_x, e)$, upon using the continuity of $ev: \mathcal{H}(X) \times X \rightarrow X$,

$$\text{we have } c - \lim_{\lim(X)} \Phi(\chi_x) = \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} \bigwedge_{x' \in X} [\lim \mathcal{G}(x') \rightarrow \lim ev^\Rightarrow(\Phi \times \mathcal{G})(\chi_x(x'))]$$

$$\leq \lime \rightarrow \lim ev^\Rightarrow(\Phi \times [e])(ev(\chi_x, e)) = \lim ev^\Rightarrow(\Phi \times [e])(ev(\chi_x, e))$$

This implies that $c - \lim_{\lim(X)} \Phi(\chi_x) \leq \lim ev^\Rightarrow(\Phi \times [e])(ev(\chi_x, e))$

$$= \lim(\chi^{-1})^{\Rightarrow}(\Phi)(\chi^{-1}(\chi_x)) \quad (3.7).$$

Hence $c - \lim_{\chi(X)} \Phi(\chi_x) \leq \lim(\chi^{-1})^{\Rightarrow}(\Phi)(\chi^{-1}(\chi_x))$, showing that

$\chi_{|\chi(X)}^{-1}: \chi(X) \rightarrow X, \chi_x \mapsto \chi^{-1}(\chi_x)$ is continuous. In fact, for any $v \in L^X$,

$$\text{we get } ev^{\Rightarrow}(\Phi \times [e])(v) = \bigvee_{v_1 \times v_2 \leq ev^{\leftarrow}(v), v_1 \in L^{\chi(X)}, v_2 \in L^X} \Phi(v_1) \wedge [e](v_2) \\ \leq \bigvee_{v_1 \times v_2 \leq ev^{\leftarrow}(v), v_1 \in L^{\chi(X)}, v_2 \in L^X} \Phi(v_1 \wedge v_2(e)) \quad (3.8).$$

Now since $v_1(\chi_x) \wedge v_2(e) \leq v(ev(\chi_x, e)) = v(\chi_x(e)) = v(x)$

$= v(\chi^{-1}(\chi_x)) = ((\chi^{-1})^{\leftarrow}(v))(\chi_x)$, we have from (3.8) that

$$(3.8) \leq \Phi((\chi^{-1})^{\leftarrow}(v)) = (\chi^{-1})^{\Rightarrow}(\Phi)(v), \text{ i.e., one obtains:}$$

$ev^{\Rightarrow}(\Phi \times [e]) \leq (\chi^{-1})^{\Rightarrow}(\Phi)$, which is used in (3.7). This ends the proof. \square

4. Construction of a stratified enriched lattice-valued convergence transformation group

Definition 4.1. Let $L = (L, \leq, \wedge)$ be a frame, $(X, \lim) \in |\text{SSL-CONV}|$, $(G, \cdot) \in |\text{GRP}|$, and $\varphi: G \times X \rightarrow X$ be a mapping satisfying the following:

- (1) $\varphi(e, x) = x$ for all $x \in X$;
- (2) $\varphi(g \cdot h, x) = \varphi(g, \varphi(h, x))$, for all $g, h \in G$ and $x \in X$;
- (3) $\varphi(g, \cdot): X \rightarrow X, x \mapsto \varphi(g, x) = \varphi_g(x)$ is continuous, for each $g \in G$.

Define $c - \lim$ on G for all $g \in G$ and for all $\Phi \in \mathcal{F}_L^s(G)$ by

$c - \lim = c - \lim \wedge c - \lim^{(-1)}$ such that

$$c - \lim \Phi(g) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim \varphi^{\Rightarrow}(\Phi \times \mathcal{F})(\varphi(g, x))] \quad (4.1)$$

$$c - \lim^{(-1)} \Phi(g) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim \varphi^{\Rightarrow}(\Phi^{-1} \times \mathcal{F})(\varphi(g^{-1}, x))] \quad (4.2)$$

Theorem 4.2. Let $L = (L, \leq, \wedge)$ be a frame. Let $(X, \lim) \in |\text{SSL-CONV}|$, and $(G, \cdot) \in |\text{GRP}|$. Then $c - \lim$ is a stratified strong L -convergence structure on G such that the triple $((G, \cdot, c - \lim), (X, \lim), \varphi) \in |\text{SSL-CONVTGRP}|$ on $(X, \lim) \in |\text{SSL-CONV}|$ with respect to φ .

Proof. We need to prove the following three items:

(i) $(G, \cdot, c - \lim) \in |\text{SSL-CONVGRP}|$;

(ii) $\varphi: G \times X \rightarrow X$ is continuous;

(iii) φ satisfies (1) and (2) above.

(iii) follows from the Definition 4.1(3). We prove (i), note that we use f, g the elements of G , and e as its identity element:

(C1) (a) Let $f \in G$ with $[f] \in \mathcal{F}_L^s(G)$, and $x \in X$. In view of (4.1), we have for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and for all $x \in X$: $c - \lim \Phi(f) \leq [\lim \mathcal{F}(x) \rightarrow \lim \varphi^{\Rightarrow}(\Phi \times \mathcal{F})(\varphi(f, x))]$.

Now due to continuity of $\varphi(g, \cdot): X \rightarrow X$, we have in particular, for any $f \in G$: $\limx \leq \lim \varphi^{\Rightarrow}([f] \times [x])(\varphi(f, x))$. But then

$$\top = \limx \rightarrow \lim \varphi^{\Rightarrow}([f] \times [x])(\varphi(f, x)) \leq c - \limf, \text{ i.e., } c - \limf = \top.$$

(b) Let $f \in G$, and since G is a group, we have $f^{-1} \in G$ and hence by continuity of $\varphi(f^{-1}, \cdot)$, in particular, one obtains:

$$\limx \leq \lim \varphi^{\Rightarrow}([f^{-1}] \times [x])(\varphi(f^{-1}, x)), \text{ and then we have } \top = \limx \rightarrow \lim \varphi^{\Rightarrow}([f^{-1}] \times [x])(\varphi(f^{-1}, x)) \leq \\ c - \lim^{(-1)}f, \text{ i.e., } c - \lim^{(-1)}f = \top. \text{ Hence combining (a) and (b), we get } c - \limf = \top.$$

(C2) (a) Let $\Phi, \Psi \in \mathcal{F}_L^s(G)$ with $\Psi \leq \Phi$. Then $\varphi^{\Rightarrow}(\Psi \times \mathcal{F}) \leq \varphi^{\Rightarrow}(\Phi \times \mathcal{F})$

which implies that $\lim \varphi^{\Rightarrow}(\Psi \times \mathcal{F}) \leq \lim \varphi^{\Rightarrow}(\Phi \times \mathcal{F})$. Then by using (4.1),

$$\text{we have for any } f \in G, c - \lim \Psi(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim \varphi^{\Rightarrow}(\Psi \times \mathcal{F})(\varphi(f, x))] \\ \leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim \varphi^{\Rightarrow}(\Phi \times \mathcal{F})(\varphi(f, x))] = c - \lim \Phi(f).$$

(b) Let $\Phi, \Psi \in \mathcal{F}_L^s(G)$ with $\Psi \leq \Phi$. Then $\varphi^{\Rightarrow}(\Psi^{-1} \times \mathcal{F}) \leq \varphi^{\Rightarrow}(\Phi^{-1} \times \mathcal{F})$

which implies that $\lim \varphi^{\Rightarrow}(\Psi^{-1} \times \mathcal{F}) \leq \lim \varphi^{\Rightarrow}(\Phi^{-1} \times \mathcal{F})$. Then using (4.2),

we get for any $f \in G$

$$c - \lim^{(-1)} \Psi(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim \varphi^{\Rightarrow}(\Psi^{-1} \times \mathcal{F})(\varphi(f^{-1}, x))]$$

$$\leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(G)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \longrightarrow \lim \varphi^{\Rightarrow} (\Phi^{-1} \times \mathcal{F})(\varphi(f^{-1}, x))] = c - \lim^{(-1)} \Phi(f).$$

Hence, combining (a) and (b), we have

$$c - \lim \Psi(f) = c - \lim \Psi(f) \wedge c - \lim^{(-1)} \Psi(f) \leq c - \lim \Phi(f) \wedge c - \lim^{(-1)} \Phi(f) \\ = c - \lim \Phi(f), \text{ i.e., } c - \lim \Psi(f) \leq c - \lim \Phi(f).$$

(C3s) Let $\Phi, \Psi \in \mathcal{F}_L^s(G)$ and $f \in G$. Then we have

$$(a) \ c - \lim \Phi(f) \wedge c - \lim \Psi(f) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(G)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \longrightarrow \lim \varphi^{\Rightarrow} (\Phi \times \mathcal{F})(\varphi(f, x))] \wedge \\ \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(G)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \longrightarrow \lim \varphi^{\Rightarrow} (\Psi \times \mathcal{F})(\varphi(f, x))] \\ \leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(G)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \longrightarrow \lim [\varphi^{\Rightarrow} (\Phi \times \mathcal{F}) \wedge \varphi^{\Rightarrow} (\Psi \times \mathcal{F})](\varphi(f, x))] \\ = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(G)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \longrightarrow \lim \varphi^{\Rightarrow} ((\Phi \wedge \Psi) \times \mathcal{F})(\varphi(f, x))] = c - \lim(\Phi \wedge \Psi)(f),$$

(b) This follows in an exact fashion as in (a) showing that

$$c - \lim^{(-1)} \Phi(f) \wedge c - \lim^{(-1)} \Psi(f) \leq c - \lim^{(-1)}(\Phi \wedge \Psi)(f).$$

Hence, combining (a) and (b), we get $c - \lim \Phi(f) \wedge c - \lim \Psi(f) \leq c - \lim(\Phi \wedge \Psi)(f)$.

(CGM) Let $\Phi, \Psi \in \mathcal{F}_L^s(G)$. Then we prove that for any $f, g \in G$:

$$c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq c - \lim(\Phi \odot \Psi)(f \cdot g). \text{ For that we proceed as follows:}$$

$$(a) \ c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq c - \lim(\Phi \odot \Psi)(f \cdot g)$$

Now in view of (4.1), $c - \lim \Phi(f) \wedge (c - \lim \Psi(g) \wedge \lim \mathcal{F}(x))$

$$\leq c - \lim \Phi(f) \wedge \lim \varphi^{\Rightarrow} (\Psi \times \mathcal{F})(\varphi(g, x))$$

$$\leq \lim \varphi^{\Rightarrow} (\Phi \times \varphi^{\Rightarrow}(\Psi \times \mathcal{F}))(\varphi(f, \varphi(g, x))), \text{ i.e.,}$$

$$c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq \lim \mathcal{F}(x) \rightarrow \lim \varphi^{\Rightarrow} (\Phi \times \varphi^{\Rightarrow}(\Psi \times \mathcal{F}))(\varphi(f, \varphi(g, x)));$$

this is true for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and for all $x \in X$, and thus,

$$c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim \varphi^{\Rightarrow} (\Phi \times \varphi^{\Rightarrow}(\Psi \times \mathcal{F}))(\varphi(f, \varphi(g, x)))].$$

Applying Proposition 2.6 along with the fact that

$$\varphi^{\Rightarrow}(\Phi \times \varphi^{\Rightarrow}(\Psi \times \mathcal{F}))(\varphi(f, \varphi(g, x))) \leq \varphi^{\Rightarrow}(\Phi \odot \Psi \times \mathcal{F})(\varphi(f \cdot g, x)),$$

$$\text{we get } c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} [\lim \mathcal{F}(x) \rightarrow \lim \varphi^{\Rightarrow} (\Phi \odot \Psi \times \mathcal{F})(\varphi(f \cdot g, x))]$$

$$= c - \lim (\Phi \odot \Psi)(f \cdot g).$$

In fact, for $v \in L^{G \times (G \times X)}$, upon manipulating the condition (2) of the Definition 4.1, i.e., $\varphi(f, \varphi(g, x)) = \varphi(f \cdot g, x)$, we have

$$\varphi^{\Rightarrow} (\Phi \times \varphi^{\Rightarrow}(\Psi \times \mathcal{F}))(v) = [\Phi \times \varphi^{\Rightarrow}(\Psi \times \mathcal{F})](\varphi^{\leftarrow}(v))$$

$$= \bigvee_{v_1 \times v_2 \leq \varphi^{\leftarrow}(v), v_1 \in L^G, v_2 \in L^{G \times X}} [\Phi(v_1) \wedge (\bigvee_{\rho_1 \times \rho_2 \leq \varphi^{\leftarrow}(v_2), \rho_1 \in L^G, \rho_2 \in L^X} \Psi(\rho_1) \wedge \mathcal{F}(\rho_2))]$$

$$= \bigvee_{v_1 \times v_2 \leq \varphi^{\leftarrow}(v); \rho_1 \times \rho_2 \leq \varphi^{\leftarrow}(v_2)} (\Phi(v_1) \wedge \Psi(\rho_1)) \wedge \mathcal{F}(\rho_2)$$

$$\leq \bigvee_{v_1(f) \wedge v_2(\varphi(g, x)) \leq v(\varphi(f, \varphi(g, x))); \rho_1(g) \wedge \rho_2(x) \leq v_2(\varphi(g, x))} (\Phi(v_1) \wedge \Psi(\rho_1)) \wedge \mathcal{F}(\rho_2)$$

$$\leq \bigvee_{v_1(f) \wedge \rho_1(g) \wedge \rho_2(x) \leq \varphi^{\leftarrow}(v)(f, g, x)} (\Phi(v_1) \wedge \Psi(\rho_1)) \wedge \mathcal{F}(\rho_2)$$

$$\leq \bigvee_{v_1(f) \wedge \rho_1(g) \wedge \rho_2(x) \leq \varphi^{\leftarrow}(v)(f, g, x)} (\bigvee_{v_1 \odot \rho_1 \leq \xi} (\Phi \odot \Psi)(\xi)) \wedge \mathcal{F}(\rho_2)$$

$$= \bigvee_{v_1(f) \wedge \rho_1(g) \wedge \rho_2(x) \leq \varphi^{\leftarrow}(v)(f, g, x); v_1(f) \wedge \rho_1(g) \leq \xi(f, g)} (\Phi \odot \Psi)(\xi) \wedge \mathcal{F}(\rho_2)$$

$$\leq \bigvee_{\xi(f, g) \wedge \rho_2(x) \leq \varphi^{\leftarrow}(v)(f, g, x)} (\Phi \odot \Psi)(\xi) \wedge \mathcal{F}(\rho_2)$$

$$= \bigvee_{\xi \times \rho_2 \leq \varphi^{\leftarrow}(v)} (\Phi \odot \Psi)(\xi) \wedge \mathcal{F}(\rho_2)$$

$$= \varphi^{\Rightarrow} (\Phi \odot \Psi \times \mathcal{F})(v), \text{ i.e., } \varphi^{\Rightarrow} (\Phi \times \varphi^{\Rightarrow}(\Psi \times \mathcal{F})) \leq \varphi^{\Rightarrow} (\Phi \odot \Psi \times \mathcal{F}).$$

(b) Following almost in the similar fashion as in (a), we can arrive at :

$$c - \lim^{(-1)} \Phi(f) \wedge c - \lim^{(-1)} \Psi(g) \leq c - \lim^{(-1)}(\Phi \odot \Psi)(f \cdot g)$$

Hence combining (a) and (b), one obtains: $c - \lim \Phi(f) \wedge c - \lim \Psi(g) \leq c - \lim \Phi \odot \Psi(f \cdot g)$.

(CGI) This follows at once from the Definition 4.1 (4.2) while for the other part

we just use the inverse of the inverse of f , and the fact that if $\Phi \in \mathcal{F}_L^s(G)$,

then $\Phi^{-1} \in \mathcal{F}_L^s(G)$.

(ii) We show that the mapping $\varphi: (G \times X, c - \lim_G \times \lim_X) \longrightarrow (X, \lim_X)$,

$(f, x) \longmapsto \varphi(f, x)$ is continuous. For, let $\Upsilon \in \mathcal{F}_L^s(G \times X)$ and $(f, x) \in G \times X$.

Then we have $(c - \lim_G \times \lim_X) \Upsilon(f, x) = c - \lim_G pr_G^{\Rightarrow}(\Upsilon)(f) \wedge \lim_X pr_X^{\Rightarrow}(\Upsilon)(x)$

$$\leq \lim_X pr_X^{\Rightarrow}(\Upsilon)(x) \rightarrow \lim_X \varphi^{\Rightarrow} (pr_G^{\Rightarrow}(\Upsilon) \times pr_X^{\Rightarrow}(\Upsilon))(\varphi(f, x)) \wedge \lim_X pr_X^{\Rightarrow}(\Upsilon)(x).$$

But as it follows from Proposition 3.6 [30] that $pr_G^{\Rightarrow}(\Upsilon) \times pr_X^{\Rightarrow}(\Upsilon) \leq \Upsilon$,

we get $(c - \lim_G \times \lim_X) \Upsilon(f, x) = c - \lim_G pr_G^{\Rightarrow}(\Upsilon)(f) \wedge \lim_X pr_X^{\Rightarrow}(\Upsilon)(x)$

$\leq \lim_X pr_X^{\Rightarrow}(\Upsilon)(x) \rightarrow \lim_X \varphi^{\Rightarrow}(\Upsilon)(\varphi(f, x)) \wedge \lim_X pr_X^{\Rightarrow}(\Upsilon)(x)$
 $\leq \lim_X \varphi^{\Rightarrow}(\Upsilon)(\varphi(f, x))$, i.e., $(c - \lim_G \times \lim_X) \Upsilon(f, x) \leq \lim_X \varphi^{\Rightarrow}(\Upsilon)(\varphi(f, x))$.
 Similarly, one can show (4.2). Hence, $c - (\lim_G \times \lim_X) \Upsilon(f, x) \leq \lim_X \varphi^{\Rightarrow}(\Upsilon)(\varphi(f, x))$.
 This ends the proof. \square

5. Examples: Continuous action of convergence approach groups on convergence approach spaces

Definition 5.1. [36, 37] A pair (X, λ) is called a convergence approach space, where the mapping $\lambda: \mathbb{F}(X) \rightarrow [0, \infty]^X$ is called a convergence approach structure on X provided:

(CAL1) $\lambda(\dot{x})(x) = 0$, where $\dot{x} = \{A \subseteq X: x \in A\}$;

(CAL2) if $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ with $\mathbb{F} \leq \mathbb{G}$, then $\lambda(\mathbb{G}) \leq \lambda(\mathbb{F})$;

(CAL3) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, $\lambda(\mathbb{F} \wedge \mathbb{G}) = \lambda(\mathbb{F}) \vee \lambda(\mathbb{G})$.

A mapping $f: (X, \lambda) \rightarrow (Y, \sigma)$ between convergence approach spaces is said to a contraction if for each $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$, $\sigma(f(\mathbb{F}))(f(x)) \leq \lambda(\mathbb{F})(x)$.

Let **CAP** denote the category whose objects are all convergence approach spaces and morphisms are all contraction mappings.

Theorem 5.2. [37] The category **CAP** is a topological construct. In particular, given a source $f_j: X \rightarrow (Y_j, \sigma_j)$, $j \in J$, the initial approach convergence structure λ on X is given for all $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$ by $\lambda(\mathbb{F})(x) = \bigvee_{j \in J} \sigma_j(f_j^{\rightarrow}(\mathbb{F}))(f_j(x))$.

Definition 5.3. [5, 38] The triple (G, \cdot, λ) is called a convergence approach group if the following are fulfilled:

(CAG1) (G, \cdot) is a group;

(CAG2) $(G, \lambda) \in |\mathbf{CAP}|$;

(CAGM) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(G)$, $x, y \in G$: $\lambda(\mathbb{F} \odot \mathbb{G})(xy) \leq \lambda(\mathbb{F})(x) \vee \lambda(\mathbb{G})(y)$;

(CAGI) $\forall \mathbb{F} \in \mathbb{F}(G)$, $x \in G$: $\lambda(\mathbb{F}^{-1})(x^{-1}) \leq \lambda(\mathbb{F})(x)$.

The category of all convergence approach groups and group homomorphisms which are contraction mappings is denoted by **CAPGRP**.

Example 5.4. [5] Let $(X, \lambda_X) \in |\mathbf{CAP}|$, and λ_Y be a convergence approach structure on a group Y . If $\mathcal{C}(X, Y) = \{h: X \rightarrow Y: f \text{ is a contraction}\}$, the set of all contraction mappings from (X, λ_X) into (Y, λ_Y) . If we define point-wise that $(fg)(x) = f(x)g(x)$ and $f^{-1}(x) = (f(x))^{-1}$ for all $f, g \in \mathcal{C}(X, Y)$ and $x \in X$, then $(\mathcal{C}(X, Y), \cdot)$ is a group and the triple $(\mathcal{C}(X, Y), \cdot, \lambda_c)$ is a convergence approach group, where $\lambda_c: \mathbb{F}(\mathcal{C}(X, Y)) \rightarrow [0, \infty]^{\mathcal{C}(X, Y)}$ is defined by $\lambda_c(\Phi)(f) = \bigwedge \mathcal{L}(\Phi, f)$ and $\mathcal{L}(\Phi, f) = \{\alpha \in [0, \infty]: \forall \mathbb{F} \in \mathbb{F}(X), x \in X, \lambda_Y(ev(\Phi \times \mathbb{F}))(f(x)) \leq \lambda_X(\mathbb{F})(x) \vee \alpha\}$, for all $\Phi \in \mathbb{F}(\mathcal{C}(X, Y))$, $f \in \mathcal{C}(X, Y)$ and $ev: \mathcal{C}(X, Y) \times X \rightarrow X$, $(f, x) \mapsto f(x)$, the evaluation mapping.

Example 5.5. [17] Let $(X, \lambda) \in |\mathbf{CAP}|$ and consider $\mathcal{H}(X) = \{f: X \rightarrow X: f \text{ is homeomorphism, i.e., } f \text{ is bijective, } f \text{ and } f^{-1} \text{ are contractions}\}$.

Define a convergence approach structure λ_c on $\mathcal{H}(X)$, the group of homeomorphism under composition: for $\Phi \in \mathcal{F}(\mathcal{H}(X))$ and $f \in \mathcal{H}(X)$ we define

$$\mathcal{L}(\Phi, f) = \{\alpha \in [0, \infty]: \forall \mathbb{F} \in \mathbb{F}(X), x \in X. \lambda(ev(\Phi \times \mathbb{F}))(f(x)) \vee \lambda(ev(\Phi^{-1} \times \mathcal{F}))(f^{-1}(x)) \leq \lambda(\mathbb{F})(x) \vee \alpha\},$$

whence $\lambda_c: \mathbb{F}(\mathcal{H}(X)) \rightarrow [0, \infty]^{\mathcal{H}(X)}$ is defined by $\lambda_c(\Phi)(f) = \bigwedge \mathcal{L}(\Phi, f)$. Then $(\mathcal{H}(X), \cdot, \lambda_c)$ is a convergence approach group.

Let $S: [0, 1] \rightarrow [0, \infty]$ be a strictly decreasing surjective mapping such that $S(1) = 0$, which is also order reversing and satisfies that $S(\bigwedge_{j \in J} \alpha_j) = \bigvee_{j \in J} S(\alpha_j)$ and $S(\bigvee_{j \in J} \alpha_j) = \bigwedge_{j \in J} S(\alpha_j)$. For this map S , there exists inverse $S^{-1}: [0, \infty] \rightarrow [0, 1]$ which is strictly decreasing and surjective, and therefore share the properties of S . We recall from [30](see also, [41]) that if $\mathcal{F} \in \mathcal{F}_{[0,1]}^s(X)$, then $\Phi_{\mathcal{F}} \in \mathbb{F}(X)$ where

$$\Phi_{\mathcal{F}} = \{A \subseteq X: \mathcal{F}(1_A) = 1\}.$$

Also, recall that if $\mathbb{F} \in \mathbb{F}(X)$, then we define the $[0, 1]$ -filter $\mathcal{F}^{\mathbb{F}}$ for any $v \in [0, 1]^X$ by

$$\mathcal{F}^{\mathbb{F}}(v) = \bigvee \{\alpha \in [0, 1]: v^\alpha \in \mathbb{F}\},$$

where $v^\alpha = \{x \in X: v(x) \geq \alpha\}$.

Proposition 5.6. [31] If $(X, \lambda) \in |\text{CAP}|$, then $(X, \lim_\lambda) \in |\text{SS}[0, 1]\text{-CONV}|$, where $\lim_\lambda(\mathcal{F})(x) = S^{-1}(\lambda(\Phi_{\mathcal{F}})(x))$, for all $\mathcal{F} \in \mathcal{F}_{[0,1]}^s(X)$.

Lemma 5.7. [31] If $f: (X, \lambda) \rightarrow (X', \lambda')$ is a contraction, then $f: (X, \lim_\lambda) \rightarrow (X', \lim_{\lambda'})$ is continuous.

Proposition 5.8. [2] If $(X, \cdot, \lambda) \in |\text{CAPGRP}|$, then $(X, \cdot, \lim_\lambda) \in |\text{SS}[0, 1]\text{-CONVGRP}|$

Lemma 5.9. Let $(G, \cdot, \lambda), (G', \cdot, \lambda') \in |\text{CAPGRP}|$ and $f: (G, \cdot, \lambda) \rightarrow (G', \cdot, \lambda')$ be a group homomorphism and contraction mapping. Then the mapping $f: (G, \cdot, \lim_\lambda) \rightarrow (G', \cdot, \lim_{\lambda'})$ is a continuous group homomorphism.

Proof. This follows at once from the Lemma 5.7 and Proposition 5.8. \square

Corollary 5.10.

$$\mathfrak{F} : \begin{cases} \text{CAPGRP} & \longrightarrow & \text{SS}[0,1]\text{-CONVGRP} \\ (X, \cdot, \lambda) & \longmapsto & (X, \cdot, \lim_\lambda) \\ f & \longmapsto & f \end{cases},$$

is a functor.

Proposition 5.11. [31] Let $(X, \lim) \in |\text{SS}[0, 1]\text{-CONV}|$. Then $(X, \lambda_{\lim}) \in |\text{CAP}|$, where $\lambda_{\lim}(\mathbb{F})(x) = S(\bigvee \{\lim \mathcal{F}(x): \Phi_{\mathcal{F}} \leq \mathbb{F}\})$.

Lemma 5.12. [31] If $f: (X, \lim) \rightarrow (X', \lim')$ is a continuous mapping, then $f: (X, \lambda_{\lim}) \rightarrow (X', \lambda_{\lim'})$ is a contraction mapping.

Proposition 5.13. Let $(X, \cdot, \lim) \in |\text{SS}[0, 1]\text{-CONVGRP}|$. Then $(X, \cdot, \lambda_{\lim}) \in |\text{CAPGRP}|$, where for any $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$ $\lambda_{\lim}(\mathbb{F})(x) = S(\bigvee \{\lim \mathcal{F}(x): \Phi_{\mathcal{F}} \leq \mathbb{F}\})$.

Proof. We only need to check the condition(CAGM): For this, let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ and $x, y \in X$, then

$$\begin{aligned} & \lambda_{\lim}(\mathbb{F})(x) \vee \lambda_{\lim}(\mathbb{G})(y) \\ &= S(\bigvee \{\lim \mathcal{F}(x): \Phi_{\mathcal{F}} \leq \mathbb{F}\}) \vee S(\bigvee \{\lim \mathcal{G}(y): \Phi_{\mathcal{G}} \leq \mathbb{G}\}) \\ &= S(\bigvee \{\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y): \Phi_{\mathcal{F}} \leq \mathbb{F}, \Phi_{\mathcal{G}} \leq \mathbb{G}\}) \\ &\geq S(\bigvee \{\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y): \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}} \leq \mathbb{F} \odot \mathbb{G}\}) \geq S(\bigvee \{\lim (\mathcal{F} \odot \mathcal{G})(xy): \Phi_{\mathcal{F} \odot \mathcal{G}} \leq \mathbb{F} \odot \mathbb{G}\}) = \lambda_{\lim}(\mathbb{F} \odot \mathbb{G})(xy), \end{aligned}$$

i.e., $\lambda_{\lim}(\mathbb{F} \odot \mathbb{G})(xy) \leq \lambda_{\lim}(\mathbb{F})(x) \vee \lambda_{\lim}(\mathbb{G})(y)$.

In fact, $A \in \Phi_{\mathcal{F} \odot \mathcal{G}} \iff \mathcal{F} \odot \mathcal{G}(1_A) = \top$ if and only if there are $v_1 := 1_{\mathcal{F}}$ and $v_2 := 1_{\mathcal{G}}$ with $1_{\mathcal{F}} \cdot 1_{\mathcal{G}} \leq 1_A$ such that $\mathcal{F}(1_{\mathcal{F}}) = \top$ and $\mathcal{G}(1_{\mathcal{G}}) = \top$, whence $F \in \Phi_{\mathcal{F}}$ and $G \in \Phi_{\mathcal{G}}$ but as it follows that $F \cdot G \subseteq A$ and as $F \cdot G \in \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}$ and since $\Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}$ is a filter yields $A \in \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}$. Hence $\Phi_{\mathcal{F} \odot \mathcal{G}} \leq \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}$. \square

Lemma 5.14. If $(X, \cdot, \lim), (X', \cdot, \lim') \in |\text{SS}[0, 1]\text{-CONVGRP}|$ and $f: (X, \cdot, \lim) \rightarrow (X', \cdot, \lim')$ is a continuous group homomorphism, then $f: (X, \cdot, \lambda_{\lim}) \rightarrow (X', \cdot, \lambda_{\lim'})$ is a group homomorphism and a contraction mapping.

Proof. In view of the Lemma 5.12 and Proposition 5.13, we are done. \square

Corollary 5.15.

$$\mathfrak{G} : \begin{cases} \text{SS}[0,1]\text{-CONVGRP} & \longrightarrow & \text{CAPGRP} \\ (X, \cdot, \lim) & \longmapsto & (X, \cdot, \lambda_{\lim}) \\ f & \longmapsto & f \end{cases},$$

is a functor.

Theorem 5.16. *CAPGRP* a reflective subcategory of *SS[0, 1]-CONVGRP*

Proof. From Corollary 5.10 and Corollary 5.15 in conjunction with the Proposition 5.5 [31](see also, Propositions 5.3 and 5.4 [31]), we obtain the embedding $\mathbf{SS}[0, 1] - \mathbf{CONVGRP} \xrightarrow{\mathfrak{G}} \mathbf{CAPGRP}$ and $\mathbf{CAPGRP} \xrightarrow{\mathfrak{F}} \mathbf{SSL} - \mathbf{CONVGRP}$ such that $\mathfrak{F} \circ \mathfrak{G} \geq id_{\mathbf{SS}[0,1]-\mathbf{CONVGRP}}$ and $\mathfrak{G} \circ \mathfrak{F} = id_{\mathbf{CAPGRP}}$. \square

Lemma 5.17. Let $(X, \cdot, \lim) \in |\mathbf{SS}[0, 1]-\mathbf{CONVGRP}|$. Then $(X, \cdot, \lambda^{\lim}) \in |\mathbf{CAPGRP}|$, where $\lambda^{\lim}(\mathbb{F})(x) = S(\lim \mathcal{F}^{\mathbb{F}}(x))$.

Proof. Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ and $x, y \in X$

$$\begin{aligned} & \lambda^{\lim}(\mathbb{F})(x) \vee \lambda^{\lim}(\mathbb{G})(y) \\ &= S(\lim \mathcal{F}^{\mathbb{F}}(x)) \vee S(\lim \mathcal{G}^{\mathbb{G}}(y)) \\ &= S(\lim \mathcal{F}^{\mathbb{F}}(x) \wedge \lim \mathcal{G}^{\mathbb{G}}(y)) \\ &\geq S(\lim(\mathcal{F}^{\mathbb{F}} \circ \mathcal{G}^{\mathbb{G}})(xy)) \\ &\geq S(\lim(\mathcal{F} \circ \mathcal{G})^{\mathbb{F} \circ \mathbb{G}}(xy)) = \lambda^{\lim}(\mathbb{F} \circ \mathbb{G})(xy). \end{aligned}$$

In fact, $\mathcal{F}^{\mathbb{F}} \circ \mathcal{G}^{\mathbb{G}}(v) = \bigvee_{v_1 \cdot v_2 \leq v} \mathcal{F}^{\mathbb{F}}(v_1) \wedge \mathcal{G}^{\mathbb{G}}(v_2)$

$$\begin{aligned} &= \bigvee_{v_1 \cdot v_2 \leq v} (\bigvee \{\alpha \in [0, 1]: v_1^\alpha \in \mathbb{F}\} \wedge \bigvee \{\beta \in [0, 1]: v_2^\beta \in \mathbb{G}\}) \\ &\leq \bigvee_{v_1 \cdot v_2 \leq v} (\bigvee \{\alpha \wedge \beta \in [0, 1]: v_1^\alpha \cdot v_2^\beta \in \mathbb{F} \circ \mathbb{G}\}) \\ &\leq \bigvee \{\gamma \in [0, 1]: v^\gamma \in \mathbb{F} \circ \mathbb{G}\} \\ &= (\mathcal{F} \circ \mathcal{G})^{\mathbb{F} \circ \mathbb{G}}; \text{ this is true, since } z \in v_1^\alpha \cdot v_2^\beta \text{ if and only if there are } x \in v_1^\alpha \text{ and } y \in v_2^\beta \text{ such that } z = xy. \text{ Now } \\ & v(z) \geq v_1 \cdot v_2(z) \geq v_1(x) \wedge v_2(y) \geq \alpha \wedge \beta \text{ which implies } z \in v^{\alpha \wedge \beta}, \text{ but as } \mathbb{F} \circ \mathbb{G} \text{ is a filter, we have } v^{\alpha \wedge \beta} \in \mathbb{F} \circ \mathbb{G}; \\ & \text{ taking } \alpha \wedge \beta = \gamma, \text{ we have } v^\gamma \in \mathbb{F} \circ \mathbb{G}. \quad \square \end{aligned}$$

Definition 5.18. Let $(X, \lambda_X) \in |\mathbf{CAP}|$ and $(G, \cdot, \lambda_G) \in |\mathbf{CAPGRP}|$. Then the triple $((G, \cdot, \lambda_G), (X, \lambda_X), \varphi)$ or in short (G, X, φ) is called a convergence approach transformation group on convergence approach space (X, λ_X) with respect to the mapping $\varphi: G \times X \rightarrow X, (g, x) \mapsto \varphi(g, x)$, if φ satisfies the conditions:

(CATG1) $\forall \mathbb{K} \in \mathbb{F}(G), \forall \mathbb{F} \in \mathbb{F}(X)$, and $\forall g \in G, \forall x \in X: \lambda_X(\varphi^{-1}(\mathbb{K} \times \mathbb{F}))(\varphi(g, x)) \leq \lambda_G(\mathbb{K})(g) \vee \lambda_X(\mathbb{F})(x)$;

(CATG2) $\varphi(e, x) = x, \forall x \in X$;

(CATG3) $\varphi(g \cdot h, x) = \varphi(g, \varphi(h, x)), \forall g, h \in G$, and $\forall x \in X$.

φ is called continuous action, G is called phase group while X is called phase space.

The category of convergence approach transformation groups denoted by **CAPTGRP** consists of all convergence approach transformation groups as objects, and all pairs $(k, f): (G, X, \varphi) \rightarrow (G', X', \varphi')$ as morphism, where

(TG1) $G \xrightarrow{k} G'$ is a **CAPGRP**-morphism, i.e., a group homomorphism and contraction mapping;

(TG2) $X \xrightarrow{f} X'$ is a **CAP**-morphism, i.e., a contraction mapping such that $\varphi' \circ (k \times f) = f \circ \varphi$.

Remark 5.19. If $(k, f): (G, X, \varphi) \rightarrow (G', X', \psi)$ and $(k', h): (G', X', \psi) \rightarrow (G'', X'', \chi)$ are morphisms of stratified enriched lattice-valued convergence transformation groups, then the composition $(k'k, hf): (G, X, \varphi) \rightarrow (G'', X'', \chi)$ is again a morphism of a stratified enriched lattice-valued convergence transformation group, where the composition is defined by $(k', h) \circ (k, f) = (k'k, hf)$. Moreover, if $G = G' = G''$ and $k = k' = id_G$, then clearly $k'k = id_G$. Similarly, one can describe the composition of morphisms for the case of convergence approach transformation groups as well as for probabilistic convergence transformation groups under t -norm $*$ as given below in Section 6.

Example 5.20. Any convergence approach group (G, \cdot, λ) can be made into a convergence approach transformation group on itself in the following way: $((G, \cdot, \lambda), (G, \lambda), \varphi)$ with respect to the mapping $\varphi: G \times X \rightarrow X$, defined by $\varphi(g, x) = gx$, where the condition (CATG1) stand as follows:

$$\forall \mathbb{K}, \mathbb{F} \in \mathbb{F}(G) \text{ and } \forall g, x \in G: \lambda(\mathbb{K} \circ \mathbb{F})(gx) \leq \lambda(\mathbb{K})(g) \vee \lambda(\mathbb{F})(x).$$

Theorem 5.21. If $((G, \cdot, \lambda_G), (X, \lambda_X), \varphi) \in |\mathbf{CAPTGRP}|$, then

$((G, \cdot, \lim_{\lambda_G}), (X, \lim_{\lambda_X}), \varphi) \in |\mathbf{SS}[0, 1]-\mathbf{CONVTGRP}|$, where $\lim_{\lambda} \mathcal{F}(x) = S^{-1}(\lambda(\Phi_{\mathcal{F}})(x))$.

Proof. We only prove the condition(CTG1): For this, we let $\mathcal{K} \in \mathcal{F}_L^s(G)$, $\mathcal{F} \in \mathcal{F}_L^s(X)$, $g \in G$ and $x \in X$. Then we have:

$$\begin{aligned} & \lim_{\lambda_G}(\mathcal{K})(g) \wedge \lim_{\lambda_X}(\mathcal{F})(x) \\ &= \mathcal{S}^{-1}(\lambda_G(\Phi_{\mathcal{K}})(g)) \wedge \mathcal{S}^{-1}(\lambda_X(\Phi_{\mathcal{F}})(x)) \\ &= \mathcal{S}^{-1}(\lambda_G(\Phi_{\mathcal{K}})(g) \vee \lambda_X(\Phi_{\mathcal{F}})(x)) \\ &\leq \mathcal{S}^{-1}(\lambda_X(\Phi_{\mathcal{K} \times \mathcal{F}})(\varphi(g, x))) \\ &\leq \mathcal{S}^{-1}(\lambda_X(\Phi_{\mathcal{K} \times \mathcal{F}})(\varphi(g, x))) \text{ (since both } \lambda_X \text{ and } \mathcal{S}^{-1} \text{ are order reversing)} \\ &= \lim_{\lambda_X}(\mathcal{K} \times \mathcal{F})(\varphi(g, x)), \text{ i.e.,} \\ & \lim_{\lambda_G}(\mathcal{K})(g) \wedge \lim_{\lambda_X}(\mathcal{F})(x) \leq \lim_{\lambda_X}(\mathcal{K} \times \mathcal{F})(\varphi(g, x)). \\ & \text{In fact, } \Phi_{\mathcal{K}} \times \Phi_{\mathcal{F}} = \{K \times F: K \in \Phi_{\mathcal{K}}, F \in \Phi_{\mathcal{F}}\} \\ &= \{K \times F: \mathcal{K}(1_K) = 1, \mathcal{F}(1_F) = 1\} \\ &\leq \{K \times F: \mathcal{K} \times \mathcal{F}(1_{K \times F}) = 1\} \\ &\leq \{H: \mathcal{K} \times \mathcal{F}(1_H) = 1\} \\ &= \Phi_{\mathcal{K} \times \mathcal{F}}. \quad \square \end{aligned}$$

Corollary 5.22.

$$\mathfrak{R} : \left\{ \begin{array}{ll} \text{CAPTGRP} & \longrightarrow \text{SS}[0,1]\text{-CONVTGRP} \\ ((G, \cdot, \lambda_G), (X, \lambda_X), \varphi) & \longmapsto ((G, \cdot, \lim_{\lambda_G}), (X, \lim_{\lambda_X}), \varphi) \\ (k, f) & \longmapsto (k, f) \end{array} \right. ,$$

is a functor.

Theorem 5.23. If $((G, \cdot, \lim_G), (X, \lim_X), \varphi) \in |\text{SS}[0, 1]\text{-CONVTGRP}|$, then $((G, \cdot, \lambda_{\lim_G}), (X, \lambda_{\lim_X}), \varphi) \in |\text{CAPTGRP}|$, where $\lambda_{\lim_G}(\mathbb{K}) = \mathcal{S}(\bigvee \{\lim_G \mathcal{K}(g) : \Phi_{\mathcal{K}} \leq \mathbb{K}\})$.

Proof. We only show that the condition (CATG1) is satisfied. To do so, let $\mathbb{K} \in \mathbb{F}(G)$, $\mathbb{F} \in \mathbb{F}(X)$, $g \in G$ and $x \in X$. Then

$$\begin{aligned} & \lambda_{\lim_G}(\mathbb{K})(g) \vee \lambda_{\lim_X}(\mathbb{F})(x) \\ &= \mathcal{S}(\bigvee \{\lim_G \mathcal{K}(g) : \Phi_{\mathcal{K}} \leq \mathbb{K}\}) \vee \mathcal{S}(\bigvee \{\lim_X \mathcal{F}(x) : \Phi_{\mathcal{F}} \leq \mathbb{F}\}) \\ &= \mathcal{S}(\bigvee \{\lim_G \mathcal{K}(g) \wedge \lim_X \mathcal{F}(x) : \Phi_{\mathcal{K}} \leq \mathbb{K}, \Phi_{\mathcal{F}} \leq \mathbb{F}\}) \\ &\geq \mathcal{S}(\bigvee \{\lim_X \varphi^{\rightarrow}(\mathcal{K} \times \mathcal{F})(\varphi(g, x)) : \Phi_{\mathcal{K} \times \mathcal{F}} \leq \mathbb{K} \times \mathbb{F}\}) \\ &= \lambda_{\lim_X}(\varphi^{\rightarrow}(\mathbb{K} \times \mathbb{F}))(\varphi(g, x)), \\ & \text{i.e., } \lambda_{\lim_X}(\varphi^{\rightarrow}(\mathbb{K} \times \mathbb{F}))(\varphi(g, x)) \leq \lambda_{\lim_G}(\mathbb{K})(g) \vee \lambda_{\lim_X}(\mathbb{F})(x). \quad \square \end{aligned}$$

In view of the previous results, and due to the references [31] and [41] we have the following:

Corollary 5.24.

$$\mathfrak{M} : \left\{ \begin{array}{ll} \text{SS}[0,1]\text{-CONVTGRP} & \longrightarrow \text{CAPTGRP} \\ ((G, \cdot, \lim_G), (X, \lim_X), \varphi) & \longmapsto ((G, \cdot, \lambda_{\lim_G}), (X, \lambda_{\lim_X}), \varphi) \\ (k, f) & \longmapsto (k, f) \end{array} \right. ,$$

is a functor.

Theorem 5.25. The category CAPTGRP is isomorphic to a reflective subcategory of the category SS[0, 1]-CONVTGRP.

Proof. This proof is almost similar to the proofs given in [31] except some algebraic parts; however, for the sake of completeness we give here an outline of the proof, and refer to [31] for the details. Remark that it follows from Theorem 5.16 that CAPGRP is a reflective subcategory of SS[0, 1]-CONVTGRP. In view of Corollary 5.22 and Corollary 5.24, and, among others, we need to see that the functor $\mathfrak{M}: \text{CAPTGRP} \rightarrow \text{SS}[0, 1]\text{-CONVTGRP}$ is injective on objects, and full.

1. To show \mathfrak{R} is injective on objects, we proceed as follows:

Let $((G, \cdot, \lambda_G), (X, \lambda_X), \varphi) \neq ((G, \cdot, \lambda'_G), (X, \lambda'_X), \psi)$, we claim that

$((G, \cdot, \lim_{\lambda_G}), (X, \lim_{\lambda_X}), \varphi) \neq ((G, \cdot, \lim_{\lambda'_G}), (X, \lim_{\lambda'_X}), \psi)$. Then first, if $\lambda_G \neq \lambda'_G$ (resp. $\lambda_X \neq \lambda'_X$), then it follows at once from Proposition 4.5 [31] that $\lim_{\lambda_G} \neq \lim_{\lambda'_G}$ (resp. $\lim_{\lambda_X} \neq \lim_{\lambda'_X}$). Secondly, if $\varphi \neq \psi$, then we immediately get that $((G, \cdot, \lim_{\lambda_G}), (X, \lim_{\lambda_X}), \varphi) \neq ((G, \cdot, \lim_{\lambda'_G}), (X, \lim_{\lambda'_X}), \psi)$.

2. To prove that \mathfrak{R} is full, we make use of the morphisms of these categories. For, let $((G, \cdot, \lambda_G), (X, \lambda_X), \varphi), ((H, \cdot, \lambda_H), (Y, \lambda_Y), \psi) \in |\mathbf{CAPTGRP}|$, and assume that $(k, f): ((G, \cdot, \lim_{\lambda_G}), (X, \lim_{\lambda_X}), \varphi) \rightarrow ((H, \cdot, \lim_{\lambda_H}), (Y, \lim_{\lambda_Y}), \psi)$ is a morphism in $\mathbf{SS}[0, 1]\text{-CONVTGRP}$, i.e., $k: G \rightarrow H$ is a $\mathbf{SS}[0, 1]\text{-CONVGRP}$ -morphism, meaning k is continuous group homomorphism while $f: X \rightarrow Y$ is a $\mathbf{SS}[0, 1]\text{-CONV}$ -morphism, meaning f is a continuous mapping such that $\psi \circ (k \times f) = f \circ \varphi$. Now due to the preceding results both of these morphisms imply that (k, f) is a morphism in $\mathbf{CAPTGRP}$, whence the group homomorphism for the case k remains the same; moreover, one can observe that $\psi \circ (k \times f) = f \circ \varphi$ holds good, where ψ, φ, k and f are all respectively contraction mappings.

Upon using Proposition 5.5 [31] in conjunction with the Theorem 5.23 and Lemma 5.14, we conclude that $\mathbf{CAPTGRP}$ is isomorphic to a reflective subcategory of $\mathbf{SS}[0, 1]\text{-CONVTGRP}$. This ends the proof. \square

We consider below another type of examples where probabilistic convergence structure is given in [25] and probabilistic convergence group in question is introduced in [3]. In what follows we consider $L = ([0, 1], \leq, *, \wedge)$ which is an enriched *cl*-premonoid.

6. Examples: Continuous action of probabilistic convergence groups on probabilistic convergence spaces under triangular norms

Definition 6.1. [25] A pair (X, \mathbf{C}) with $\mathbf{C} = (c_x)_{x \in X}$, where $c_x: \mathbb{F}(X) \rightarrow [0, 1]$ is called a probabilistic convergence space under *t*-norm $*$ if and only if the following conditions are fulfilled:

(PC1) $\forall x \in X: c_x(\dot{x}) = 1$;

(PC2) $\forall x \in X, \forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ with $\mathbb{F} \leq \mathbb{G}$ implies $c_x(\mathbb{F}) \leq c_x(\mathbb{G})$;

(PC3) $\forall x \in X, \forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X), c_x(\mathbb{F}) * c_x(\mathbb{G}) \leq c_x(\mathbb{F} \wedge \mathbb{G})$.

A mapping $f: (X, \mathbf{C}) \rightarrow (X', \mathbf{C}')$ between probabilistic convergence spaces (X, \mathbf{C}) and (X', \mathbf{C}') is called continuous if and only if for all $x \in X$ and for all $\mathbb{F} \in \mathbb{F}(X), c_x(\mathbb{F}) \leq c_{f(x)}(f(\mathbb{F}))$.

The category of probabilistic convergence spaces under *t*-norm $*$, and continuous mappings between them is denoted by \mathbf{PCONV}^* .

Definition 6.2. [3] A triple (X, \cdot, \mathbf{C}) is called a probabilistic convergence group under *t*-norm $*$ if and only if the following are true:

(PCG1) $(X, \cdot) \in |\mathbf{GRP}|$;

(PCG2) $(X, \mathbf{C}) \in |\mathbf{PCONV}^*|$;

(PCGM) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X) \forall x, y \in X: c_x(\mathbb{F}) * c_y(\mathbb{G}) \leq c_{xy}(\mathbb{F} \odot \mathbb{G})$;

(PCGI) $\forall x \in X, \forall \mathbb{F} \in \mathbb{F}(X): c_x(\mathbb{F}) \leq c_{x^{-1}}(\mathbb{F}^{-1})$.

The category of all probabilistic convergence groups under *t*-norm $*$ and continuous group homomorphisms is denoted by $\mathbf{PCONVGRP}^*$.

Lemma 6.3. [41] \mathbf{PCONV}^* is a isomorphic to a full subcategory of $\mathbf{S}[0, 1]\text{-CONV}$.

Proposition 6.4. [3] If (X, \cdot, \mathbf{C}) is a probabilistic convergence group under *t*-norm $*$, then $(X, \cdot, \lim_{\mathbf{C}})$ is a stratified $[0, 1]$ -convergence group, where $\lim_{\mathbf{C}}(\mathcal{F})(x) = c_x(\Phi_{\mathcal{F}})$, for any $\mathcal{F} \in \mathcal{F}_{[0,1]}^s(X)$ and $x \in X$.

Lemma 6.5. [41] \mathbf{PCONV}^* is a isomorphic to a reflective subcategory of $\mathbf{S}[0, 1]\text{-CONV}$.

Proposition 6.6. If (X, \cdot, \lim) is a stratified $[0, 1]$ -valued convergence group, then $(X, \cdot, \mathbf{C}_{\lim})$ is a probabilistic convergence group under *t*-norm $*$, where $c_x(\mathbb{F}) = \bigvee \{\lim \mathcal{F}(x) : \Phi_{\mathcal{F}} \leq \mathbb{F}\}$.

Proof. In view of Lemma 6.5, we only need to prove (PCGM).

For, let $x, y \in X$ and $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$. Then we have

$$\begin{aligned} & c_x(\mathbb{F}) * c_y(\mathbb{G}) \\ &= \bigvee_{\Phi_{\mathcal{F}} \leq \mathbb{F}} \lim \mathcal{F}(x) * \bigvee_{\Phi_{\mathcal{G}} \leq \mathbb{G}} \lim \mathcal{G}(y) \\ &= \bigvee_{\Phi_{\mathcal{F}} \leq \mathbb{F}, \Phi_{\mathcal{G}} \leq \mathbb{G}} \lim \mathcal{F}(x) * \lim \mathcal{G}(y) \\ &\leq \bigvee_{\Phi_{\mathcal{F} \circ \mathcal{G}} \leq \Phi_{\mathcal{F}} \circ \Phi_{\mathcal{G}} \leq \mathbb{F} \circ \mathbb{G}} \lim \mathcal{F} \circ \mathcal{G}(xy) \text{ (by Definition 3.3(CGM))} \\ &= \bigvee_{\Phi_{\mathcal{F} \circ \mathcal{G}} \leq \mathbb{F} \circ \mathbb{G}} \lim \mathcal{F} \circ \mathcal{G}(xy) \\ &= c_{xy}(\mathbb{F} \circ \mathbb{G}). \text{ That is, } c_x(\mathbb{F}) * c_y(\mathbb{G}) \leq c_{xy}(\mathbb{F} \circ \mathbb{G}). \text{ Condition (PCGI) follows immediately from the continuity condition in Lemma 6.5. } \square \end{aligned}$$

Due to preceding results we have the following:

Corollary 6.7.

$$\mathfrak{P} : \begin{cases} S[0,1]\text{-CONVGRP} & \longrightarrow & \text{PCONVGRP}^* \\ (X, \cdot, \lim) & \longmapsto & (X, \cdot, \mathbb{C}_{\lim}) \\ f & \longmapsto & f \end{cases},$$

and

$$\mathfrak{R} : \begin{cases} \text{PCONVGRP}^* & \longrightarrow & S[0,1]\text{-CONVGRP} \\ (X, \cdot, \mathbb{C}) & \longmapsto & (X, \cdot, \lim_{\mathbb{C}}) \\ f & \longmapsto & f \end{cases},$$

are functors.

Proof. Objects wise correspondence can be seen from Proposition 6.4 and Proposition 6.6 in conjunction with Lemma 6.5 while morphisms are clearly true since in Lemma 6.3 and 6.5 these are already proved, whence the case for group homomorphisms are also true. \square

Theorem 6.8. PCONVGRP^* is isomorphic to a reflective subcategory of $S[0, 1]\text{-CONVGRP}$.

Proof. In view of Lemma 6.3, the functor $\mathfrak{R} : \text{PCONVGRP}^* \longrightarrow S[0, 1]\text{-CONVGRP}$ is injective on objects and full. Then it follows upon using Lemma 6.3 that the category PCONVGRP^* is isomorphic to a full subcategory of $S[0, 1]\text{-CONVGRP}$. To prove the reflectivity remark that due to Lemma 6.9 [41] in conjunction with the Proposition 6.6, one can show that the functors in Corollary 6.7:

$$S[0, 1]\text{-CONVGRP} \xrightarrow{\mathfrak{P}} \text{PCONVGRP}^* \text{ and } \text{PCONVGRP}^* \xrightarrow{\mathfrak{R}} S[0, 1]\text{-CONVGRP}$$

yield that $\mathfrak{R} \circ \mathfrak{P} \geq id_{S[0,1]\text{-CONVGRP}}$, and $\mathfrak{P} \circ \mathfrak{R} = id_{\text{PCONVGRP}^*}$.

Hence we infer that PCONVGRP^* is isomorphic to a reflective subcategory of $S[0, 1]\text{-CONVGRP}$. \square

Definition 6.9. Let $(X, \mathbb{C}) \in |\text{PCONV}^*|$ and $(G, \cdot, \mathbb{C}) \in |\text{PCONVGRP}^*|$. Then the triple $((G, \cdot, \mathbb{C}), (X, \mathbb{C}), \succ)$ or in short (G, X, \succ) is called a probabilistic convergence transformation group under t -norm $*$ on probabilistic convergence space (X, \mathbb{C}) under t -norm $*$ with respect to the mapping $\succ : G \times X \longrightarrow X, (g, x) \longmapsto \succ(g, x)$, if \succ satisfies the conditions:

(PCTG1) $\forall \mathbb{K} \in \mathbb{F}(G), \forall \mathbb{T} \in \mathbb{F}(X)$, and $\forall g \in G, \forall x \in X: c_g(\mathbb{K}) * c_x(\mathbb{T}) \leq c_{\succ(g,x)}(\succ \rightarrow (\mathbb{K} \times \mathbb{T}))$;

(PCTG2) $\succ(e, x) = x, \forall x \in X$;

(CATG3) $\succ(g \cdot h, x) = \succ(g, \succ(h, x)), \forall g, h \in G$, and $\forall x \in X$.

\succ is called continuous action, G is called phase group while X is called phase space.

The category of probabilistic convergence transformation groups under t -norm $*$ is denoted by PCONVTGRP^* consists of all probabilistic convergence transformation groups under t -norm $*$ as objects, and all pairs $(k, f) : (G, X, \succ) \longrightarrow (G', X', \succ')$ as morphism, where

(PCTG1) $G \xrightarrow{k} G'$ is a PCONVGRP^* -morphism, i.e., a group homomorphism and continuous mapping;

(PCTG2) $X \xrightarrow{f} X'$ is a PCONV^* -morphism, i.e., a continuous mapping such that $\succ' \circ (k \times f) = f \circ \succ$.

Theorem 6.10. *If $((G, \cdot, \mathbf{C}_G), (X, \mathbf{C}_X), \lambda) \in |\mathbf{PCONVTGRP}^*|$, then $((G, \cdot, \lim_{\mathbf{C}_G}), (X, \lim_{\mathbf{C}_X}), \lambda) \in |\mathbf{S}[0, 1]\text{-CONVTGRP}|$.*

Proof. It suffices to prove the condition (CTG1). For, let $\mathcal{F} \in \mathcal{F}_{[0,1]}^s(G)$ and $\mathcal{G} \in \mathcal{F}_{[0,1]}^s(X)$, and $(g, x) \in G \times X$.

Then we have

$$\begin{aligned} & \lim_{\mathbf{C}_G} \mathcal{F}(g) * \lim_{\mathbf{C}_X} \mathcal{G}(x) \\ &= c_g(\Phi_{\mathcal{F}}) * c_x(\Phi_{\mathcal{G}}) \\ &\leq c_{\lambda(g,x)}(\lambda^{\rightarrow}(\Phi_{\mathcal{F}} \times \Phi_{\mathcal{G}})) \\ &\leq c_{\lambda(g,x)}(\Phi_{\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G})}) \\ &= \lim_{\mathbf{C}_X}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}))(\lambda(g, x)). \text{ That is, we have} \\ & \lim_{\mathbf{C}_G} \mathcal{F}(g) * \lim_{\mathbf{C}_X} \mathcal{G}(x) \leq \lim_{\mathbf{C}_X}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}))(\lambda(g, x)). \quad \square \end{aligned}$$

Corollary 6.11.

$$\mathfrak{Y} : \begin{cases} \mathbf{PCONVTGRP}^* & \longrightarrow & \mathbf{S}[0,1]\text{-CONVTGRP} \\ ((G, \cdot, \mathbf{C}_G), (X, \mathbf{C}_X), \lambda) & \longmapsto & ((G, \cdot, \lim_{\mathbf{C}_G}), (X, \lim_{\mathbf{C}_X}), \lambda) \text{ ,} \\ (k, f) & \longmapsto & (k, f) \end{cases}$$

is a functor.

Theorem 6.12. *If $((G, \cdot, \lim_G), (X, \lim_X), \lambda) \in |\mathbf{S}[0, 1]\text{-CONVTGRP}|$, then $((G, \cdot, \mathbf{C}_{\lim_G}), (X, \mathbf{C}_{\lim_X}), \lambda) \in |\mathbf{PCONVTGRP}^*|$.*

Proof. We only need to prove the condition (PCTG1). In order to do so, let $\mathbf{K} \in \mathbf{F}(G), \mathbf{T} \in \mathbf{F}(X), (g, x) \in G \times X$, then we have

$$\begin{aligned} & c_g(\mathbf{K}) * c_x(\mathbf{T}) \\ &= \bigvee_{\Phi_{\mathbf{K}} \leq \mathbf{K}} \lim_G \mathcal{K}(g) * \bigvee_{\Phi_{\mathbf{G}} \leq \mathbf{T}} \lim_X \mathcal{G}(x) \\ &= \bigvee_{\Phi_{\mathbf{K}} \leq \mathbf{K}, \Phi_{\mathbf{G}} \leq \mathbf{T}} \lim_G \mathcal{K}(g) * \lim_X \mathcal{G}(x) \\ &\leq \bigvee_{\Phi_{\mathbf{K}} \leq \mathbf{K}, \Phi_{\mathbf{G}} \leq \mathbf{T}} \lim_X(\lambda^{\Rightarrow}(\mathcal{K} \times \mathcal{G}))(\lambda(g, x)) \\ &\leq \bigvee_{\Phi_{\mathbf{K} \times \mathbf{G}} \leq \Phi_{\mathbf{F}} \times \Phi_{\mathbf{G}} \leq \mathbf{K} \times \mathbf{T}} \lim_X(\lambda^{\Rightarrow}(\mathcal{K} \times \mathcal{G}))(\lambda(g, x)) \\ &\leq \bigvee_{\Phi_{\mathbf{K} \times \mathbf{G}} \leq \mathbf{K} \times \mathbf{T}} \lim_X(\lambda^{\Rightarrow}(\mathcal{K} \times \mathcal{G}))(\lambda(g, x)) \\ &= c_{\lambda(g,x)}(\lambda^{\Rightarrow}(\mathbf{K} \times \mathbf{T})) \quad \square \end{aligned}$$

Considering the morphisms as described in the Definition 6.9 and their composition in the light of the Remark 5.19, the Theorem 6.12 yields the following:

Corollary 6.13.

$$\mathfrak{Y} : \begin{cases} \mathbf{S}[0,1]\text{-CONVTGRP} & \longrightarrow & \mathbf{PCONVTGRP}^* \\ ((G, \cdot, \lim_G), (X, \lim_X), \lambda) & \longmapsto & ((G, \cdot, \mathbf{C}_{\lim_G}), (X, \mathbf{C}_{\lim_X}), \lambda) \text{ ,} \\ (k, f) & \longmapsto & (k, f) \end{cases}$$

is a functor.

Theorem 6.14. *$\mathbf{PCONVTGRP}^*$ is isomorphic to a reflective subcategory of $\mathbf{S}[0, 1]\text{-CONVTGRP}$.*

Proof. It is pointed out in Theorem 6.8 that $\mathbf{PCONVTGRP}^*$ is isomorphic to a reflective subcategory of $\mathbf{S}[0, 1]\text{-CONVTGRP}$; upon using Lemma 6.5 ([41]: Lemma 6.9), Theorem 6.12, and following almost similar proof as in Theorem 5.25, and noting that the morphism (k, f) in $\mathbf{S}[0, 1]\text{-CONVTGRP}$ implies that (k, f) is a morphism in $\mathbf{PCONVTGRP}^*$ such that $\lambda' \circ (k \times f) = f \circ \lambda$. Thus, we can conclude that $\mathbf{PCONVTGRP}^*$ is isomorphic to a reflective subcategory of $\mathbf{S}[0, 1]\text{-CONVTGRP}$. \square

7. Conclusion

In this article, we have discussed the action of enriched lattice-valued convergence groups providing the action of convergence approach groups as well as action of probabilistic convergence groups under t -norm $*$, as natural examples. In doing so, we have considered $L = (L, \leq, *, \wedge)$, an enriched cl -premonoid, and when dealing with the function space equipped with lattice-valued continuous convergence structure [30], we have taken $L = (L, \leq, \wedge)$, a frame [22, 49] or a complete Heyting algebra. However, in the present scenario, we have achieved a good number of interesting results, leaving the cases for $L = (L, \leq, *, \otimes)$, an arbitrary enriched cl -premonoid lattice, still an open problem. There are some interesting points worth mentioning that we intend to address in a separate paper: one, to look into these works from monoidal categorical perspective [26, 51], also this is one of the suggestions made by one of the referees; second, at present we considered the mapping $S: [0, 1] \rightarrow [0, \infty]$ in Sections 5 and 6 as a continuation of our previous work in [3] is originated from the papers [31] and [41] that we strictly followed hereof. But as mentioned by one of the referees that the mapping $S: [0, 1] \rightarrow [0, \infty]$ could be avoided following an alternative route, i.e., the examples of Section 5 can be captured by taking $L = \{0, 1\}$ for filters and taking $M = ([0, \infty], \geq, +)$ (the extended half-line, opposite order, addition as quantale operation) for the other part; and the examples in Sections 6 by taking $L = \{0, 1\}$ for the filters, and $M = ([0, 1], \leq, *)$ with a continuous t -norm $*$ for other side; in this way, use of the mapping $S: [0, 1] \rightarrow [0, \infty]$ could be simplified. Thanks to the referee for raising this interesting point, of which, unfortunately, we were not aware of, will certainly be looked into while working in a paper devoted to monoidal categories as noted above. Furthermore, we will be interested in future to see if there is any relationship between probabilistic convergence transformation groups that we studied in [4], and the enriched lattice-valued convergence transformation groups that we considered herein this text; although, apparently these two approaches are different, given the fact that both the generalizations inherited their root from the work of Park [42]. However, we do not rule out the possibility of a connection between the examples given in Section 6 and in [4] (see also [6]), but these are beyond the scope of the present paper, and will be dealt with in one of our future articles.

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