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Almost Ricci solitons isometric to spheres

Sharief Deshmukh

Department of Mathematics
College of Science, King Saud University
P. O. Box-2455, Riyadh 11451, Saudi Arabia
shariefd@ksu.edu.sa

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We find a characterization of a sphere using a compact gradient almost Ricci soliton and the lower bound on the integral of Ricci curvature in the direction of potential field. Also, we use Poisson equation on a compact gradient almost Ricci soliton to find a characterization of the unit sphere.

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1. Introduction

A Ricci soliton on a Riemannian manifold (M, g) is a stationary solution of the Ricci flow equation (cf. [4, 5])

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(t), \tag{1}$$

of the form $g(t) = \sigma(t)\varphi_t^*$ with initial condition $g(0) = g$, where $\mathrm{Ric}(t)$ is the Ricci tensor of the evolving metric $g(t)$, φ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function. Equivalently, a stationary solution of Eq. (1) is given by (M, g, ξ, λ), where ξ is the vector field satisfying

$$\mathrm{Ric} + \frac{1}{2}\pounds_\xi g = \lambda g,$$

λ is a constant called soliton constant. It is a natural generalization of Einstein metrics and is a widely studied topics in Riemannian geometry (cf. [4] for a review).

Recently, Pigola *et al.* (cf. [8]), introduced the notion of an almost Ricci soliton by allowing the soliton constant λ in Ricci soliton to be a smooth function. Thus, an almost Ricci soliton is a special solution of the Ricci flow equation (1), of the form $g(t) = \sigma(t, x)\varphi_t^*$, where the scaling function $\sigma(t, x)$ is not only function of time t but also of the coordinates of point x and the initial conditions $g(0) = g$,

$\varphi_0 =$ identity, that is, $\sigma(0, x) = 1$. In this case, the special solution of the evolution equation (1) is the almost Ricci soliton (M, g, ξ, f), where the vector field ξ is called the potential field and the smooth function f is called the soliton function and it satisfies

$$\mathrm{Ric} + \frac{1}{2}\pounds_\xi g = fg. \tag{2}$$

If the function f is a constant, then the almost Ricci soliton is a Ricci soliton (cf. [4]). Therefore, we call an almost Ricci soliton (M, g, ξ, f) with non-constant soliton function a nontrivial almost Ricci soliton. Moreover, if the potential vector field ξ is a gradient of a smooth function, then the almost Ricci soliton (M, g, ξ, f) is called a gradient almost Ricci soliton.

Recently, geometry of almost Ricci soliton is subject of interest and have been taken up by many geometers and interesting results have been obtained on this subject (cf. [1–3, 9, 10]). In [3], as well as in [10], authors have shown that a compact nontrivial almost Ricci soliton with constant scalar curvature is a gradient almost Ricci soliton and that it is isometric to a sphere $\mathbb{S}^n(c)$.

Recall that finding a characterizations of spheres is an important question in geometry of Riemannian manifolds and a natural question arises whether an almost Ricci soliton structure on a compact Riemannian manifold could be utilized in getting a characterization of spheres. In this paper, we show that an appropriate lower bound on the integral of Ricci curvature of a compact gradient almost Ricci soliton (M, g, ξ, f) in the direction of the potential field ξ gives a characterization of the sphere $\mathbb{S}^n(c)$ (cf. Theorem 1).

Also, using Eq. (2), we see that the divergence of the potential field ξ of an almost Ricci soliton (M, g, ξ, f) satisfies $\operatorname{div} \xi = nf - S$, where S is the scalar curvature of the almost Ricci soliton, $n = \dim M$. This allows us to consider the Poisson equation $\Delta\sigma = S - nf$ on a compact almost Ricci soliton (M, g, ξ, f) (cf. [7]). It is known that this Poisson equation has a unique solution up to a constant on the compact almost Ricci soliton (M, g, ξ, f) (cf. [7]). Does this Poisson equation on a compact almost Ricci soliton impacts its geometry? This question could lead to several consequential studies. In this paper, we prove that the soliton function f of an n-dimensional compact gradient almost Ricci soliton (M, g, ξ, f) with Ricci curvature in the direction of the potential field ξ bounded above by $(n-1)$ is a solution of the Poisson equation $\Delta\sigma = S - nf$, if and only if it is isometric to the unit sphere $\mathbb{S}^n$, (cf. Theorem 2).

2. Preliminaries

Let (M, g, ξ, f) be an almost Ricci soliton and $\mathfrak{X}(M)$ be the Lie algebra of smooth vector fields on M. We denote by ∇_X, $X \in \mathfrak{X}(M)$ the covariant derivative with respect to the vector field X. Then, the curvature tensor field of almost Ricci soliton (M, g, ξ, f) is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M)$$

and the Ricci tensor Ric of (M, g, ξ, f) is given by

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame, $n = \dim M$. The Ricci tensor Ric is a symmetric tensor and we have the Ricci operator Q defined by $\mathrm{Ric}(X, Y) = g(QX, Y)$, which is a symmetric operator. The scalar curvature S of the almost Ricci soliton (M, g, ξ, f) is given by $S = TrQ$ (Trace of the operator Q) and the gradient ∇S of the scalar curvature S satisfies

$$\frac{1}{2}\nabla S = \sum_{i=1}^{n} (\nabla Q)(e_i, e_i), \tag{3}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame and the covariant derivative $(\nabla Q)(X, Y) = \nabla_X QY - Q(\nabla_X Y)$.

Let η be the smooth 1-form dual to potential field ξ. Define a skew symmetric operator φ on M by

$$\frac{1}{2} d\eta(X, Y) = g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{4}$$

Then using Eqs. (2) and (4) in Koszul's formula (cf. [5]), the covariant derivative of the potential field ξ is given by

$$\nabla_X \xi = fX - Q(X) + \varphi X, \quad X \in \mathfrak{X}(M). \tag{5}$$

Using Eq. (5), we compute the curvature tensor of the almost Ricci soliton (M, g, ξ, f) and get

$$\begin{aligned} R(X, Y)\xi = {} & X(f)Y - Y(f)X - (\nabla Q)(X, Y) + (\nabla Q)(Y, X) \\ & + (\nabla \varphi)(X, Y) - (\nabla \varphi)(Y, X). \end{aligned} \tag{6}$$

Taking trace in above equation, and using symmetry of the operator Q, the skew symmetry of operator φ and Eq. (3), we conclude that

$$Q(\xi) = -(n-1)\nabla f + \frac{1}{2}\nabla S - \sum_{i=1}^{n} (\nabla \varphi)(e_i, e_i), \tag{7}$$

where $n = \dim M$. Using the skew symmetry of the operator φ and Eq. (5), we compute the divergence $\operatorname{div} \xi$ of the potential field ξ to get

$$\operatorname{div} \xi = (nf - S). \tag{8}$$

The squared norm $\|Q\|^2$ of the Ricci operator Q is given by

$$\|Q\|^2 = \sum_{i=1}^{n} g(Qe_i, Qe_i),$$

and consequently, we get

$$\left|\mathrm{Ric} - \frac{S}{n} g\right|^2 = \|Q\|^2 - \frac{S^2}{n}, \tag{9}$$

where

$$|\mathrm{Ric}|^2 = \sum_{i,j=1}^{n} (\mathrm{Ric}(e_i, e_j))^2.$$

In [3], it is shown that on an n-dimensional compact almost Ricci soliton (M, g, ξ, f), the following holds:

$$\int_M \left|\mathrm{Ric} - \frac{S}{n} g\right|^2 = \frac{n-2}{2n} \int_M g(\nabla S, \xi). \tag{10}$$

3. Characterizing Spheres Using Almost Ricci Solitons

In this section, we find characterizations of spheres using an n-dimensional compact almost Ricci soliton (M, g, ξ, f).

Theorem 1. *Let (M, g, ξ, f), be an n-dimensional compact nontrivial gradient almost Ricci soliton with scalar curvature S, $n > 2$. Then the Ricci curvature in the direction of the potential vector field ξ*

$$\int_M \mathrm{Ric}(\xi, \xi) \geq \frac{n-1}{n} \int_M (nf - S)^2,$$

if and only if (M, g, ξ, f) is isometric to the sphere $\mathbb{S}^n(c)$ for a positive constant c.

Proof. Suppose (M, g, ξ, f) is an n-dimensional compact nontrivial gradient almost Ricci soliton that satisfies

$$\int_M \mathrm{Ric}(\xi, \xi) \geq \frac{n-1}{n} \int_M (nf - S)^2. \tag{11}$$

Then, the soliton function f is not a constant and $\varphi = 0$ (ξ being gradient of a function, is closed). Consequently, Eq. (5), takes the form

$$\nabla_X \xi = fX - QX, \tag{12}$$

which gives

$$\|\nabla \xi\|^2 = nf^2 + \|Q\|^2 - 2fS. \tag{13}$$

Also, Eq. (2), gives

$$\frac{1}{4}|\pounds_\xi|^2 = nf^2 + |\mathrm{Ric}|^2 - 2fS. \tag{14}$$

On a compact Riemannian manifold (M, g), we have the following integral formula (cf. [6, 10]):

$$\int_M \left(\mathrm{Ric}(\xi, \xi) + \frac{1}{2}|\pounds_\xi|^2 - \|\nabla \xi\|^2 - (\operatorname{div} \xi)^2\right) = 0.$$

Now, using Eq. (12), we have $\operatorname{div} \xi = (nf - S)$, inserting this and the values from Eqs. (13) and (14), in above equation, we conclude

$$\int_M (\mathrm{Ric}(\xi, \xi) + nf^2 + \|Q\|^2 - 2fS - (nf - S)^2) = 0,$$

which can be re-arranged as

$$\int_M \left(\left(\mathrm{Ric}(\xi,\xi) - \frac{n-1}{n}(nf-S)^2\right) + \left(\|Q\|^2 - \frac{S^2}{n}\right)\right) = 0.$$

Using the inequality (11) and the Schwarz inequality in above equation, we get the following equalities

$$Q = \frac{S}{n}I, \quad \mathrm{Ric}(\xi,\xi) = \frac{n-1}{n}(nf-S)^2. \tag{15}$$

Since, $n > 2$, the first equation gives, S is a constant (compare with Eq. (3)). Thus, Eq. (12) yields,

$$\nabla_X \xi = \left(f - \frac{S}{n}\right) X, \tag{16}$$

which gives the following expression for the curvature tensor:

$$R(X,Y)\xi = X(f)Y - Y(f)X,$$

and consequently, the following expression for Ricci tensor

$$\mathrm{Ric}(Y,\xi) = -(n-1)Y(f).$$

The above equation, implies

$$Q(\xi) = -(n-1)\nabla f,$$

which together with the first equation in Eq. (15), gives

$$\frac{S}{n}\xi = -(n-1)\nabla f.$$

Letting $\rho = \frac{S}{n} - f$ and noticing that f is not a constant, we see that ρ is not a constant and that

$$\frac{S}{n}\xi = (n-1)\nabla\rho,$$

which in view of Eq. (12), gives

$$(n-1)\nabla_X \nabla\rho = -\frac{S}{n}\rho X, \tag{17}$$

where we used Eq. (16). The Laplacian of the function ρ, $\Delta\rho$ is given by $\Delta\rho = \mathrm{div}(\nabla\rho)$, which on using Eq. (17), is computed as

$$\Delta\rho = -\frac{S}{(n-1)}\rho.$$

Note that ρ being not a constant, above equation implies ρ is an eigenfunction of the Laplace operator and consequently, $S > 0$. Hence, as Eq. (17) is Obata's differential equation, (M,g,ξ,f) is isometric to the sphere $S^n(c)$, where $S = n(n-1)c$.

Conversely, consider the sphere $\mathbb{S}^n(c)$ as hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ with unit normal vector field N. Then, the shape operator A of the sphere

$\mathbb{S}^n(c)$ is given by $A = -\sqrt{c}I$. We choose a nonzero constant vector field Z on the Euclidean space $\mathbb{R}^{n+1}$, and denote by ξ, the tangential projection of Z to $\mathbb{S}^n(c)$ and define a smooth function $\rho = \langle Z, N\rangle$. Then, we have $Z = \xi + \rho N$, which on taking covariant derivative with respect to a vector field X on $\mathbb{S}^n(c)$ and using Gauss and Weingarten formulae for hypersurface, we conclude

$$0 = \nabla_X \xi - \sqrt{c} g(X, \xi)N + X(\rho)N + \sqrt{c}\rho X,$$

where g is the induced metric on the hypersurface $\mathbb{S}^n(c)$. Equating tangential and normal components, we get

$$\nabla_X \xi = -\sqrt{c}\rho X, \quad \nabla\rho = \sqrt{c}\xi. \tag{18}$$

Thus, computing the Lie derivative $\pounds_\xi g$, we have

$$(\pounds_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X), \quad X, Y \in \mathfrak{X}(M),$$

which in view of Eq. (18), yields

$$\pounds_\xi g = -2\sqrt{c}\rho g. \tag{19}$$

The Ricci tensor of the hypersurface $\mathbb{S}^n(c)$ is given by $\mathrm{Ric} = (n-1)cg$ and consequently, using Eq. (19), we reach at

$$\frac{1}{2}\pounds_\xi g + \mathrm{Ric} = fg, \tag{20}$$

where $f = ((n-1)c - \sqrt{c}\rho)$. Note that the function ρ is not a constant, for if ρ is a constant, then Eq. (18), gives $\operatorname{div}\xi = -n\sqrt{c}\rho$ and its integration gives $\rho = 0$. Thus, we get $\nabla_X \xi = 0$ and it will imply that the sectional curvature of $\mathbb{S}^n(c)$, $R(X, \xi; \xi, X) = 0$, with the consequence that $\xi = 0$, which amounts to, $Z = 0$, which is contrary to the assumption that Z is a nonzero vector field. Hence, f is a not a constant function and consequently, by Eq. (20), we get that $(\mathbb{S}^n(c), g, \xi, f)$ is a gradient almost Ricci soliton, where $\xi = \frac{1}{\sqrt{c}}\nabla\rho$. Moreover, Eq. (18), gives $\Delta\rho = -nc\rho$, which implies

$$\int_{\mathbb{S}^n(c)} \|\nabla\rho\|^2 = nc\int_{\mathbb{S}^n(c)} \rho^2. \tag{21}$$

Note that, $nf - S = n(n-1)c - n\sqrt{c}\rho - n(n-1)c = -n\sqrt{c}\rho$, and that

$$\left(\frac{n-1}{n}\right)(nf - S)^2 = n(n-1)c\rho^2.$$

Using above equation in Eq. (21), we conclude that

$$\int_{\mathbb{S}^n(c)} \mathrm{Ric}(\nabla\rho, \nabla\rho) = \frac{n-1}{n}c\int_{\mathbb{S}^n(c)} (nf - S)^2.$$

Inserting $\nabla\rho = \sqrt{c}\xi$, in above equation, gives the required condition. $\square$

Next, we note that by virtue of Eq. (8), on a compact almost Ricci soliton (M, g, ξ, f), the following Poisson equation (cf. [7])

$$\Delta\sigma = (S - nf)$$

admits a unique solution up to a constant. We use this Poisson equation to prove the following characterization of the unit sphere $\mathbb{S}^n$:

Theorem 2. *Let (M, g, ξ, f), be an n-dimensional compact nontrivial gradient almost Ricci soliton with scalar curvature S, $n > 2$, with Ricci curvature in the direction of potential field ξ bounded above by $(n-1)$ Then the soliton function f is a solution of the Poisson equation*

$$\Delta\sigma = S - nf,$$

if and only if (M, g, ξ, f) is isometric to the unit sphere $\mathbb{S}^n$.

Proof. Suppose (M, g, ξ, f) is a compact almost Ricci soliton with Ricci curvature satisfying

$$\mathrm{Ric}(\xi, \xi) \le (n-1)\|\xi\|^2 \tag{22}$$

and that the soliton function f is a solution of the Poisson equation $\Delta\sigma = S - nf$, that is,

$$\Delta f = S - nf. \tag{23}$$

Since, (M, g, ξ, f) is a gradient almost Ricci soliton, we have $\xi = \nabla h$ for a smooth function h. Using Eq. (8), we conclude

$$-\Delta h = S - nf. \tag{24}$$

Now, as the solution of the Poisson equation is unique up to a constant, we have $-h = f + c$ for a constant c, and consequently, $\xi = -\nabla f$ holds. We compute the Hessian operator A_f of the soliton function

$$A_f X = \nabla_X \nabla f = -\nabla_X \xi = -fX + QX,$$

where we used Eq. (12). Above expression for the Hessian operator A_f, gives

$$\|A_f\|^2 = nf^2 + \|Q\|^2 - 2fS,$$

which in view of Eq. (9), implies

$$\|A_f\|^2 = \left|\mathrm{Ric} - \frac{S}{n}g\right|^2 + \frac{1}{n}(S - nf)^2. \tag{25}$$

Now, we use the following Bochner's formula:

$$\int_M (\mathrm{Ric}(\nabla f, \nabla f) + \|A_f\|^2 - (\Delta f)^2) = 0,$$

which on using $\xi = -\nabla f$ and Eqs. (23) and (25), gives

$$\int_M \left(\mathrm{Ric}(\xi, \xi) + \left|\mathrm{Ric} - \frac{S}{n}g\right|^2 + \frac{1}{n}(S - nf)\Delta f - (S - nf)\Delta f\right) = 0.$$

The above equation implies

$$\int_M \left(\operatorname{Ric}(\xi,\xi) + \left|\operatorname{Ric} - \frac{S}{n}g\right|^2 - \frac{n-1}{n}S\Delta f + (n-1)f\Delta f\right) = 0.$$

Integrating the last term in above equation by parts, gives

$$\int_M \left(\operatorname{Ric}(\xi,\xi) - (n-1)\|\nabla f\|^2 + \left|\operatorname{Ric} - \frac{S}{n}g\right|^2 - \frac{n-1}{n}S\Delta f\right) = 0.$$

Now, using $\operatorname{div}(S\nabla f) = g(\nabla f, \nabla S) + S\Delta f = -\xi(S) + S\Delta f$ in above equation, we conclude

$$\int_M \left(\operatorname{Ric}(\xi,\xi) - (n-1)\|\xi\|^2 + \left|\operatorname{Ric} - \frac{S}{n}g\right|^2 - \frac{n-1}{n}g(\xi,\nabla S)\right) = 0,$$

which in view of Eq. (10), gives

$$\int_M \left(\operatorname{Ric}(\xi,\xi) - (n-1)\|\xi\|^2 - \frac{n}{n-2}\left|\operatorname{Ric} - \frac{S}{n}g\right|^2\right) = 0.$$

Using inequality (22) in above equation, we conclude that

$$\operatorname{Ric}(\xi,\xi) = (n-1)\|\xi\|^2 \quad \text{and} \quad \operatorname{Ric} = \frac{S}{n}g.$$

These equations imply $S = n(n-1)$ and therefore Eq. (12) takes the form

$$\nabla_X \nabla f = ((n-1) - f)X,$$

which gives the following Obata's differential equation:

$$\nabla_X \nabla \overline{f} = -\overline{f}X,$$

where $\overline{f} = (n-1) - f$ is a not a constant function owing to the fact that (M, g, ξ, f) is a nontrivial almost Ricci soliton. Hence, (M, g, ξ, f) is isometric to the unit sphere $\mathbb{S}^n$.

The converse follows as in Theorem 1, on taking $c = 1$. □

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